MODULAR THEORY OF GROUP CHARACTERS.

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1. THE problem of the representation of a given finite group as a linear homogeneous group with real or complex coefficients has been fully treated by Frobenius * by means of his theory of group characters. The present paper and the companion paper to appear simultaneously in the *Transactions* give a first attack on the corresponding problem for linear congruence groups, and in general for finite linear groups in any field F having a prime modulus p. To obtain simple results, it is in general necessary to introduce certain irrationalities, viz., roots of equations with coefficients in F. As our reference field we shall take the field F_p composed of the totality of integral rational functions with integral coefficients of all Galois imaginaries of all degrees, *i. e.*, the roots of congruences irreducible modulo p. In other words, F_p is the aggregate of the Galois fields $GF[p^n]$, $n = 1, 2, 3, \cdots$. Hence every equation with coefficients in F_p is completely solvable in F_p.

The paper also gives a report on the various expositions of the algebraic theory from the standpoint of their availability in the treatment of the modular theory (cf. §§ 3, 5, but particularly § 13).

2. *Definitions.* Given a finite group H with the h elements $H_0, H_1, \cdots, H_{h-1}$, we shall say that the h matrices of degree f (or linear substitutions)

$$A_{H_i} = (a_{\alpha\beta}^{H_i})_{\alpha,\beta=1,\cdots,f} \quad (i = 0, 1, \cdots, h-1), \tag{1}$$

whose elements a are marks of the field F_p, define a representation of the group H if the matrices satisfy the h^2 relations

$$A_R A_S = A_{RS} \quad (R, S = H_0, \cdots, H_{h-1}). \tag{2}$$

The matrices need not be distinct, so that the isomorphism may be multiple. Let $x_{H_i} (i = 0, \cdots, h-1)$ be independent variables. Then

$$X = \sum_R A_R x_R \quad (R = H_0, \cdots, H_{h-1}) \tag{3}$$

is called the group matrix corresponding to the representation.

* *Berliner Sitzungsberichte*, from 1896 to date.

If B is a matrix of degree f whose elements are marks of F_p and of determinant not zero, then $B^{-1}XB$ is also a group matrix, called equivalent to X. A group matrix X is called reducible or irreducible (in F_p), according as it is or is not equivalent to a matrix $\left(\begin{smallmatrix} Y & 0 \\ W & Z \end{smallmatrix}\right)$, where Y and Z are square matrices, W a rectangular matrix.

The system of h marks $\chi(H_0), \cdots, \chi(H_{h-1})$, defined by

$$\chi(R) = \sum_{a=1}^{f} a_{aa}^{R} \qquad (R = H_0, \cdots, H_{h-1}), \tag{4}$$

is called a character of the group H for the field F_p, corresponding to the particular representation (1) or group matrix (3). If the latter is irreducible, the character is called simple.

Analogous to the multiplication table of H, the matrix

$$(x_{PQ^{-1}}) \qquad (P, Q = H_0, \cdots, H_{h-1}), \tag{5}$$

in which P indicates the row and Q the column, is called the regular group matrix.

3. For the reasons to be pointed out in § 13, the only one of the various expositions of Frobenius's theory which may be utilized in the construction of a corresponding general modular theory is that by I. Schur, "Neue Begründung der Theorie der Gruppencharaktere," *Berliner Sitzungsberichte*, March 23, 1905. In Frobenius's theory, matrix (5) is completely reducible (with zero matrices to the right and left of the irreducible diagonal matrices), and each irreducible factor occurs to a power equal to its degree; in the modular theory these theorems do not hold true when p divides the order h of the group. In the latter case, only the earlier part of Schur's work can be utilized for the modular theory and then only after essential modifications.

The developments by Schur, pages 409–411, as well as the auxiliary theorems from the general theory of matrices, are valid for our field F_p. His theorem I is valid in any field; his theorem II, however, is valid only in a field F which, like F_p or the field of all real and complex numbers, has the property that every equation with coefficients in F is solvable in F. Instead of his fundamental theorem IV, we have the following (valid whether or not p divides f):

THEOREM. *If in the field F_p X and X' are non-equivalent irreducible group matrices of degrees f and f',*

(6) $X=(x_{\alpha\beta}), x_{\alpha\beta}=\sum_R a^R_{\alpha\beta}x_R (\alpha,\beta=1,\cdots,f;\ R=H_0,\cdots,H_{h-1})$,

(7) $X'=(x'_{\alpha\beta}), x'_{\alpha\beta}=\sum_R b^R_{\alpha\beta}x_R (\alpha,\beta=1,\cdots,f';\ R=H_0,\cdots,H_{h-1})$,

then the following relations hold:

(I) $\qquad \sum_R a^{R^{-1}}_{\alpha\beta}a^R_{\gamma\delta}=ce_{\alpha\delta}e_{\beta\gamma} \qquad (\alpha,\beta,\gamma,\delta=1,\cdots,f)$,

(II) $\qquad \sum_R a^{R^{-1}}_{\alpha\beta}b^R_{\gamma\delta}=0 \qquad (\alpha,\beta=1,\cdots,f;\ \gamma,\delta=1,\cdots,f')$,

in which $e_{\alpha\delta}=0$ *if* $\alpha\neq\delta$, $e_{\alpha\alpha}=1$, *while* c *is a constant for a given group matrix* X, *equal to that for any equivalent group matrix, and*

(8) $\qquad fc=h.$

In view of Schur's proof we have (II) and

(I_1) $\qquad \sum_R a^{R^{-1}}_{\alpha\beta}a^R_{\gamma\delta}=c_{\beta\gamma}e_{\alpha\delta}$,

in which the $c_{\beta\gamma}$ are constants to be determined. In view of (1), relations (2) are equivalent to

(9) $\sum_{\beta=1}^{f} a^R_{\alpha\beta}a^S_{\beta\delta}=a^{RS}_{\alpha\delta} \quad (\alpha,\delta=1,\cdots,f;\ R,S=H,\cdots,H_{h-1})$.

In (I_1), replace α by ρ, γ by σ; then multiply by $a^S_{\alpha\rho}a^T_{\gamma\sigma}$ and sum for $\rho,\sigma=1,\cdots,f$. We get

$$\sum_R\Big(\sum_\rho a^S_{\alpha\rho}a^{R^{-1}}_{\rho\beta}\Big)\Big(\sum_\sigma a^T_{\gamma\sigma}a^R_{\sigma\delta}\Big)=\sum_{\rho,\sigma}e_{\rho\delta}c_{\beta\sigma}a^S_{\alpha\rho}a^T_{\gamma\sigma}.$$

Applying (9) we get

(10) $\qquad \sum_R a^{SR^{-1}}_{\alpha\beta}a^{TR}_{\gamma\delta}=a^S_{\alpha\delta}\sum_\sigma c_{\beta\sigma}a^T_{\gamma\sigma}.$

For $T=E=$ identity, we get, since $a^E_{\gamma\sigma}=e_{\gamma\sigma}$ by Schur's proof,

(11) $\qquad \sum_R a^{SR^{-1}}_{\alpha\beta}a^R_{\gamma\delta}=a^S_{\alpha\delta}c_{\beta\gamma}.$

In (10) replace $c_{\beta\sigma}$ by its value from (I_1) for $\alpha=\delta$, $\gamma=\sigma$. The right member of (10) becomes

$$a^S_{\alpha\delta}\sum_\sigma a^T_{\gamma\sigma}\Big(\sum_R a^{R^{-1}}_{\alpha\beta}a^R_{\sigma\alpha}\Big)=a^S_{\alpha\delta}\sum_R a^{R^{-1}}_{\alpha\beta}a^{TR}_{\gamma\alpha},$$

in view of (9). In (11) replace R by R^{-1}, S by T, γ by α, δ by β, α by γ, β by α. We get

$$\sum_R a_{\alpha\beta}^{R^{-1}} a_{\gamma\alpha}^{TR} = a_{\gamma\beta}^{T} c_{\alpha\alpha}.$$

We have now shown that (10) becomes

$$\text{(III)} \qquad \sum_R a_{\alpha\beta}^{SR^{-1}} a_{\gamma\delta}^{TR} = a_{\alpha\delta}^{S} a_{\gamma\beta}^{T} c_{\alpha\alpha}.$$

In (III) set $T = E$ and compare the result with (11). We get

$$\text{(12)} \qquad a_{\alpha\delta}^{S} c_{\gamma\beta} c_{\alpha\alpha} = a_{\alpha\delta}^{S} c_{\beta\gamma} \quad (\alpha, \beta, \gamma, \delta = 1, \cdots, f).$$

If for α fixed, the $a_{\alpha\delta}^{S}$ were all zero, then by (6) every $x_{\alpha\beta} = 0$, *i. e.*, all the elements of the αth row of X would vanish. But $|X|$ is not identically zero. Hence, by (12),

$$c_{\beta\gamma} = c_{\beta\gamma} c_{\alpha\alpha} \qquad (\alpha, \beta, \gamma = 1, \cdots, f).$$

In particular, the $c_{\alpha\alpha}$ are all equal. Call their common value c. Then

$$\text{(13)} \qquad c_{\beta\gamma} = c c_{\beta\gamma}.$$

Hence (I_1) becomes (I). In (II) replace α by ρ, γ by σ; then multiply by $a_{\alpha\rho}^{S} b_{\gamma\sigma}^{T}$ and sum for ρ, σ. We get

$$\text{(IV)} \qquad \sum_R a_{\alpha\beta}^{SR^{-1}} b_{\gamma\delta}^{TR} = 0.$$

Now (I), for $\delta = \alpha$, $\gamma = \beta$, becomes

$$\text{(14)} \qquad c = \sum_R a_{\alpha\beta}^{R^{-1}} a_{\beta\alpha}^{R}.$$

Summing for β, applying (9) and $a_{\alpha\alpha}^{E} = 1$, we obtain (8).

Finally, we prove that $c = d$, d being the constant for

$$P^{-1}XP = \sum_R (d_{\alpha\beta}^{R}) \mathfrak{x}_R.$$

Let $P = (p_{\alpha\beta})$, $P^{-1} = (q_{\alpha\beta})$, so that

$$S_\delta \equiv \sum_\sigma q_{\delta\sigma} p_{\sigma\delta} = 1.$$

Then

$$d_{\gamma\delta}^{R} = \sum_{\rho,\sigma} q_{\gamma\rho} a_{\rho\sigma}^{R} p_{\sigma\delta},$$

$$d \equiv \sum_R d_{\gamma\delta}^{R^{-1}} d_{\delta\gamma}^{R} = \sum_{\rho,\sigma,\tau,\delta} q_{\gamma\rho} p_{\sigma\delta} q_{\delta\tau} p_{\tau\gamma} \Big(\sum_R a_{\rho\sigma}^{R^{-1}} a_{\tau\delta}^{R} \Big).$$

The final term equals $ce_{\rho\sigma}e_{\sigma\tau}$ by (I). Hence

$$d = c\Big(\sum_{\rho} q_{\gamma\rho}p_{\rho\gamma}\Big)\Big(\sum_{\sigma} q_{\delta\sigma}p_{\sigma\delta}\Big) = cS_\gamma S_\delta = c.$$

4. Certains relations between the simple characters (4) follow readily from (I)–(IV) and (9). In (III) take $\beta = \alpha$, $\delta = \gamma$, and sum for α, γ. Hence

$$\text{(V)} \qquad \sum_R \chi(SR^{-1})\chi(TR) = c\chi(ST).$$

For $T =$ identity, (V) becomes

$$\text{(VI)} \qquad \sum_R \chi(SR^{-1})\chi(R) = c\chi(S).$$

Conversely, replacing S by ST and R by RT in (VI), we get (V). In fact, by (4) and (9), we find that

$$\text{(VII)} \qquad \chi(RT) = \chi(TR), \quad \chi(E) = f \quad (E = \text{identity}).$$

For $S =$ identity, (VI) becomes, by (8),

$$\text{(VIII)} \qquad \sum_R \chi(R^{-1})\chi(R) = h.$$

In (III) take $\gamma = \beta$, $\delta = \alpha$, sum for α, β and apply (9). Hence

$$\text{(IX)} \qquad c\chi(S)\chi(T) = \sum_R \chi(SR^{-1}TR).$$

Let χ' be the character corresponding to the irreducible group matrix X' not equivalent to X. In (IV) set $\beta = \alpha$, $\delta = \gamma$, and sum for α, γ. Hence

$$\text{(X)} \qquad \sum_R \chi(SR^{-1})\chi'(TR) = 0.$$

If the modulus p does not divide h, relations (VI) and (X) follow* from (VII), (VIII), (IX) (the latter three with $c = h/f$ serve in the algebraic theory to determine completely the characters).

Further, when p is prime to h, the entire exposition by Schur is valid in the field F_p. In particular the algebraically irreducible factors of the regular group determinant have integral

* Cf. Weber, Algebra, 2d ed., II, p. 194.

algebraic numbers as coefficients and hence are functions interpretable in F_p and are irreducible in F_p. The fact that the modular theory for p prime to h is identical with the algebraic theory was first stated in the writer's paper, *Transactions*, volume 3 (1902), page 285.

5. For a given group H of order h, let $X, X', \cdots$ be group matrices of degrees $f, f', \cdots$, irreducible in the field F_p, and no two equivalent. Give them the notation (6), (7), etc. Then the $f^2+f'^2+\cdots$ linear functions $x_{\alpha\beta}, x'_{\alpha\beta}, \cdots$ are linearly independent in F_p.

In case p does not divide h, the proof follows at once from relations (I) and (II) (cf. Schur, page 412). In the contrary case the proof fails, since c vanishes for certain group matrices. We may, however, make use of theorem I in the paper by Frobenius and Schur on the equivalence of linear groups, *Berliner Sitzungsberichte*, 1906, page 209, the proof being valid in our field F_p.

It follows that the determinant of an irreducible group matrix X is an irreducible function of the variables x_R; and that two irreducible group matrices are equivalent if and only if their determinants are identically equal.

Let Φ denote the determinant $|X|$ of the irreducible group matrix (3) of degree f. If E is the identity element of the group, and $R \neq E$, the coefficient of $x_E^{f-1}x_R$ in Φ is $\chi(R)$, in view of (4) and $a_{\alpha\beta}^E = e_{\alpha\beta}$. Also $\chi(E) = f$ by (VII). Hence for every element R, $\chi(R)$ is the coefficient of x_E^{f-1} in $\partial\Phi/\partial x_R$. This property is taken as the definition of $\chi(R)$ in Frobenius's second exposition, *Sitzungsberichte*, 1896, page 1349. The latter paper gives a method of determining all the coefficients of Φ in terms of the characters $\chi(R)$. The method must be modified in the case of a modular field. If u is a new independent variable, set

$$(15) \quad \Phi(x_E+u, x_A, x_B, \cdots) = u^f + \Phi_1 u^{f-1} + \cdots + \Phi_f,$$

where Φ_n is an integral homogeneous function of the nth degree of $x_E, x_A, \cdots$, and $\Phi_f = \Phi$. Let S_n denote the sum of the nth powers of the negatives of the quantities $u_1, \cdots, u_f$ for which (15) vanishes. By Frobenius's proof,

$$S_1 = \Phi_1 = \sum_R \chi(R)x_R, \quad S_n = \sum_{R_1, \cdots, R_n} \chi(R_1 \cdots R_n)x_{R_1} \cdots x_{R_n},$$

where $R_1, \cdots, R_n$ range independently over the h elements of

H. By Waring's formula each Φ_i is expressed as a polynomial in the S_s. For a modular field, this formula is not always applicable in view of the denominators. We shall illustrate for small values of f a valid method of determining in F_p the function

$$(16) \qquad \Phi = \sum C_{R_{i_1}, \cdots, R_{i_f}} x_{R_{i_1}} \cdots x_{R_{i_f}} \qquad (i_1 \leqq i_2 \leqq \cdots \leqq i_f).$$

For $f = 2$, Φ is determined by

$$(17) \qquad 2\Phi_2 = S_1^2 - S_2,$$

unless $p = 2$. By comparing the coefficients of $x_E y_E x_R y_S$ $(R \neq E, S \neq E)$ in

$$(18) \qquad \Phi(z) = \Phi(x)\Phi(y), \quad z_T = \sum_R x_R y_{R^{-1}T},$$

we obtain,* according as $R \neq S$ or $R = S$,

$$(19) \quad C_{RS} = \chi(R)\chi(S) - \chi(RS), \quad 2C_{RR} = \chi(R)\chi(R) - \chi(R^2).$$

For modulus $p = 2$, it remains to determine C_{RR}. Now

$$-S_3 + \Phi_1 S_2 - \Phi_2 S_1 = 0,$$

by Newton's identities. The coefficients of

$$x_E^3, \; x_E^2 x_S \quad (R \neq S \neq E),$$

give

$$C_{RR}\chi(R) = \chi(R)\chi(R^2) - \chi(R^3),$$

$$C_{RR}\chi(S) = -C_{RS}\chi(R) - 3\chi(R^2S) + \chi(R^2)\chi(S) + 2\chi(RS)\chi(R).$$

But C_{RS} is given by (19_1). Hence C_{RR} is determined by the characters unless the latter all vanish. But in that case Φ involves only the squares of the variables x_R and hence is reducible in F_2, contrary to hypothesis.

For $f = 3$, Φ is determined by

$$(20) \qquad 6\Phi_3 = S_1^3 - 3S_1S_2 + 2S_3$$

unless $p = 2$ or 3. For $p = 3$, we treat (20) as an algebraic identity and find that the terms in $x_R^2 x_S$ and $x_R x_S x_T$ are multiples of 3; we thus obtain C_{RRS} and C_{RST} algebraically in terms of the characters. In

*Or directly, by treating (17) as an algebraic identity, one half the coefficients of $x_R x_S (R \neq S)$ are given by (19_1).

$$S_4 - \Phi_1 S_3 + \Phi_2 S_2 - \Phi_3 S_1 = 0, \tag{21}$$

we select the terms x_R^4 and $x_R^3 x_S$, using (17), and obtain $C_{RRR}\chi(R)$ and $C_{RRR}\chi(S)$ as functions of the characters. But if every character vanishes, Φ involves only the cubes of the variables and is reducible modulo 3.

For $f = 3$, $p = 2$, Newton's identities give

$$S_3 + S_1^3 = \Phi_3 + \Phi_2 S_1, \quad \Phi_2(S_3 + S_1^3) = S_5 + S_1^2 S_3. \tag{22}$$

In Φ_2, C_{RS} is given by (19_1); to find C_{RR} we examine the coefficients of $x_R^i x_S^{5-i}$ $(i = 5, \cdots, 2)$ in (22_2). If in each the factor of C_{RR} vanishes then $S_2 + S_1^3 = 0$, identically. But Φ_3 would then be reducible by (22_1). Hence Φ_2 and then Φ_3 can be expressed in terms of the characters.

6. THEOREM. *In the field F_p any commutative group H of order $h = p^\pi q$ (q not divisible by the prime p) has exactly q distinct characters.*

If A is an element of period p^t of H, and we set $\chi(A) = W$, then in the field F_p we have

$$W^{p^t} = 1, \quad (W-1)^{p^t} = 1, \quad W = 1.$$

Proceeding as in Weber's Algebra, II, Chapter 2, we conclude that H has at most $h/p^\pi = q$ different characters. But at least this number occur, since there exist in F_p primitive roots of $x^k = 1$ for k prime to p. In fact, $x^k = 1$ has no double root and hence k distinct roots in F_p.

7. THEOREM. *If C denotes the commutator group of H, the number of distinct linear factors in F_p of the group determinant of H is obtained by dividing the order of the quotient group $Q = H/C$ by the highest power of p dividing the order of Q.*

The proof follows from §6 and the argument in Frobenius's paper, *Sitzungsberichte*, 1896, pages 1347–1349. As there shown the linear factors occur to the same power. But the argument showing that this power is the first is not valid in F_p (cf. §8).

8. THEOREM. *If p^π is the highest power of p which divides the order of a group H, every irreducible factor, in the field F_p, of the regular group determinant of H enters to a power which is an exact multiple of p^π. For any character of degree f, $\chi(P) = f$ if P is an element of period a power of p.*

The proof is given in the July number of the *Transactions*.

9. The characters in the field F_p of a group H whose order

h is prime to p may be taken to be the algebraic characters given by Frobenius's theory. We shall consider examples in which h is divisible by p. If H is abelian or if h is a power of p, the characters follow from §§ 6, 8.

The relations obtained in § 4 do not in general completely determine the simple characters in F_p. Relation (IX) is equivalent to that obtained by replacing S and T by any elements conjugate to them or by interchanging S and T. Similarly for (VI) and (X); in the latter it suffices to set $T =$ identity. Not all the values of a simple character are zero (§ 5). If $\Phi = \Phi'\Phi''$ then $\chi(R) = \chi'(R) + \chi''(R)$ for every R.

10. Let H be the symmetric group on three letters. Let E, A, B be elements of period 1, 3, 2, and set $\chi(E) = f$, $\chi(A) = a$, $\chi(B) = b$. Then, by § 4,

$$f^2 + 2a^2 + 3b^2 = 6, \quad a^2 + 2af + 3b^2 = ca, \quad 4ab + 2fb = cb,$$

$$ca^2 = 3a + 3f, \quad cab = 6b, \quad cb^2 = 2f + 4a.$$

For modulus 2, the solutions (other than $f = a = b = 0$) are

$$(23) \quad f_1 = a_1 = b_1 = 1, \quad c_1 = 0; \quad f_2 = b_2 = 0, \quad a_2 = c_2 = 1.$$

Hence by § 8, $|H| = \Phi_1^2\Phi_2^2$. Here Φ_2 is irreducible (end of § 9).

For modulus 3, the conditions reduce to $f = a$, $c = 0$. But by §§ 7, 8, $|H| = \Phi_1^3\Phi_1'^3$.

11. Let H be the alternating group on four letters. Let f, x, y, z be the values of a character when the arguments are non-conjugate elements of period 1, 2, 3, 3. Then, by § 4,

$$(24) \quad f^2 + 3x^2 + 8yz = 12, \quad 2fx + 2x^2 + 8yz = cx,$$

$$(25) \quad 2fy + 6xy + 4z^2 = cy, \quad 2fz + 6xz + 4y^2 = cz,$$

$$(26) \quad 4f + 8x = cx^2, \quad 12y = cxy, \quad 12z = cxz,$$

$$(27)_1 \quad 3f + 9x = cyz, \quad 12z = cy^2, \quad 12y = cz^2.$$

Let first the modulus p be 3. The solutions are

$$(28) \quad f = y = z = 0, \ x = \pm 1, \ c = \mp 1; \quad f = x = y = z \neq 0, \ c = 0.$$

By § 5 or § 8, $f = 1$ for the second, $f = 3$ for the first. Further, the lower signs must be taken. Thus $|H| = \Phi_1^3\Phi_3^3$.

For $p = 2$, the relations reduce to $c = 0$, $x = f$. By §§ 7, χ, we have $|H| = \Phi_1^4\Phi_1'^4\Phi_1''^4$. It follows from (17) that $8^2(R) \equiv \chi(R^2)$, whence $x = f$, $y^2 = z$, $z^2 = y$.

12. Finally, let H be the alternating group on five letters. There are five sets of conjugate elements, of periods 1, 3, 2, 5, 5. Denote the corresponding values of $\chi(R)$ by f, A, B, C, D. By § 4,

$$f^2 + 20A^2 + 15B^2 + 12C^2 + 12D^2 = 60, \tag{29}$$

$$2fA + 7A^2 + 12AB + 6AC + 6AD + 3B^2 + 6BC + 6BD + 3C^2 + 6CD + 3D^2 = cA, \tag{30}$$

$$2fC + 5A^2 + 10AB + 10AC + 10AD + 5B^2 + 10BD + 5C^2 + 2CD + D^2 = cC, \tag{31}$$

$$2(fB + 4A^2 + 4AB + 4AC + 4AD + 4BC + 4BD + B^2 + 4CD) = cB, \tag{32}$$

$$3(f + 7A + 6B + 3C + 3D) = cA^2, \tag{33}$$

$$15(A + B + C + D) = cAC, \tag{34}$$

$$3(8A + 4B + 4C + 4D) = cAB, \tag{35}$$

$$20(A + B + D) = cBC, \quad 5(f + 5A + 5C + D) = cC^2, \tag{36}$$

$$4(f + 4A + 2B + 4C + 4D) = cB^2, \quad 5(5A + 5B + C + D) = cCD, \tag{37}$$

together with the relations derived from (31), (34) and (36) by interchanging C with D. Now Φ_f must equal, or be an irreducible factor in F_p, of one of the algebraically irreducible factors of the group determinant

$$|H| = \Phi_1\Phi_3^3\Phi_3'^3\Phi_4^4\Phi_5^5.$$

Let $p = 5$. If $c \neq 0$, we obtain

$$f = C = D = 0, \quad A = \pm 1, \quad B = \mp 1, \quad c = \pm 3. \tag{38}$$

For $c = 0$, the conditions (29)–(37) reduce to

$$f = C = D, \quad B = 3f + 3A, \quad c = 0. \tag{39}$$

In (38) $f = 5$, since $f < 10$ by § 5. The only algebraically irreducible factor of degree $\geqq 5$ has $f = 5$, $C = D = 0$, $A = -1$, $B = 1$. Hence the lower signs must hold in (38). If the resulting function Φ_5 were reducible, $\Phi_5 = \Pi\Phi_\rho$, the characters

of the irreducible factors would satisfy (39) and $\sum A_i = A$, etc. As this is seen to be impossible, Φ_5 is irreducible in F_5. The remaining irreducible factors of $|H|$ satisfy (39) and have $f \leqq 4$. But the algebraically irreducible factor Φ_4 occurs only to the fourth power; hence by § 8 it is reducible in F_5. Now $\chi_4(R) \equiv \chi_1(R) + \chi_3(R)$. Hence $\Phi_4 = \Phi_1\Phi_3$ in F_5. Finally, there is no factor Φ_2, since H cannot be represented as a binary linear group in F_5, the only binary transformation of period 2 being $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence the algebraic factor Φ_3 remains irreducible in F_5. Hence

$$|H| = \Phi_1^5\Phi_3^{10}\Phi_5^5 \text{ in } F_5. \tag{40}$$

Let next $p = 3$. If $c \neq 0$, we obtain

$$f = A = 0,\ c = B = \pm 1,\ C \text{ and } D \text{ roots of } y^2 \pm y - 1 = 0. \tag{41}$$

If $c = 0$, the conditions (29)–(37) reduce to

$$f = A,\ D = C,\ A + B + C = 0,\ c = 0. \tag{42}$$

Proceeding as for $p = 5$, we find that the lower signs must hold in (41) and that the resulting functions Φ_3 and Φ_3' are irreducible in F_3; also that $\Phi_5 = \Phi_1\Phi_4$. Thus

$$|H| = \Phi_1^6\Phi_3^3\Phi_3'^3\Phi_4^9 \text{ in } F_3. \tag{43}$$

Finally, let $p = 2$. If $c \neq 0$, we obtain

$$f = B = 0,\ A = C = D = 1,\ c = 1. \tag{44}$$

If $c = 0$, conditions (29)–(37) reduce to

$$f = B, \quad A + B + C + D = 0. \tag{45}$$

By the argument employed when $p = 5$, we find that the algebraically irreducible function Φ_4, whose characters are congruent to those in (44), remains irreducible in F_2. Since H is simply isomorphic with the group of all binary transformations of determinant unity in the $GF[2^2]$, there exist factors Φ_2 irreducible in F_2. Their characters are

$$f = B = 2, \quad A = 1, \quad C \text{ and } D \text{ roots of } y^2 + y + 1 = 0. \tag{46}$$

We find that in F_2,

$$\Phi_5 = \Phi_1\Phi_2\Phi_2', \quad \Phi_3 = \Phi_1\Phi_2, \quad \Phi_3' = \Phi_1\Phi_2'.$$

Hence

$$|H| = \Phi_1^{12}\Phi_2^8\Phi_2'^8\Phi_4^4 \text{ in } F_2. \tag{47}$$

13. Frobenius's first method of introducing group characters (*Sitzungsberichte*, 1896, page 985) relates primarily to the factorization, into linear factors, of the special group determinant $|x_{PQ^{-1}}|$, in which $x_A = x_B$ if A and B are conjugate elements of the group H. This method is not suitable for the foundation of the modular theory; in fact, two distinct irreducible factors of the general group determinant G may correspond in the field F_p to the same linear factor of the special group determinant S. Thus if H is the symmetric group on three letters, and $p = 3$, $S = (x_E - x_A)^6$, while G has two distinct factors (§ 7 or § 10). It may be argued indirectly (cf. end of § 4) that the prime factors of Frobenius's fundamental determinant $|p_{\alpha\beta}|$ are all divisors of the order of H.

Frobenius's second method (l. c., page 1343) relates initially to the general group determinant G. With the exceptions noted in §§ 5, 7, the developments in his first three sections are valid in the field F_p. But his derivation in § 5 of the relations between the characters is not valid in F_p, when p divides the order h of the group. In fact, the power e to which any irreducible factor occurs in G is a multiple of p (§ 7), so that $\partial\Phi^e/\partial x_R$ vanishes identically in F_p. His method, when applied to F_p, gives rise only to relations all of whose coefficients vanish.

Burnside's first treatment, *Proceedings London Mathematical Society*, volume 29 (1898), pages 207–224, 546–565, depends upon the special group determinant. Although his auxiliary theorems were established by purely rational processes by the writer in the *Transactions*, volume 3 (1902), page 285, it was indicated in the last paper, that the method is limited to fields whose modulus does not divide the order of the group.

Burnside's second treatment, *Acta Mathematica*, volume 28 (1904), page 369, and *Proceedings London Mathematical Society*, volume 1 (1901), page 117, depends upon the existence of an invariant Hermitian form, and hence is not applicable in the construction of a general theory of modular groups.

The exposition by Schur has the advantage that after the modifications noted in § 3, it yields important results in the modular theory.

THE UNIVERSITY OF CHICAGO,
May, 1907.

ON THE SHORTEST DISTANCE BETWEEN CONSECUTIVE STRAIGHT LINES.

BY MR. JOSEPH LIPKE.

CERTAIN well-known geometric results concerning space curves and surfaces have been obtained by a discussion of the shortest distance between consecutive positions of a straight line moving continuously in space. These results have been gained by a discussion (recapitulated in §1) of the *numerator* of the expression for the shortest distance.* It is the purpose of this paper to complete the discussion by examining (§§ 2–3) the *denominator* of the distance expression, placing special emphasis upon the conditions that the distance be an infinitesimal of the *second* order, and upon a geometric interpretation of this case.

§ 1. *Brief Discussion of the Numerator.*

The equations of the straight line are

$$(1)\qquad \begin{aligned} x &= az + p, \\ y &= bz + q, \end{aligned} \quad \text{or} \quad \begin{aligned} x &= a(t)z + p(t), \\ y &= b(t)z + q(t), \end{aligned}$$

where a, b, p, q are analytic functions of a single variable t. We define the consecutive line by the equations

$$(2)\qquad \begin{aligned} x &= a(t + dt)z + p(t + dt), \\ y &= b(t + dt)z + q(t + dt), \end{aligned}$$

or

$$x = \left(a + a'dt + a''\frac{dt^2}{2!} + \cdots\right)z + \left(p + p'dt + p''\frac{dt^2}{2!} + \cdots\right),$$

$$y = \left(b + b'dt + b''\frac{dt^2}{2!} + \cdots\right)z + \left(q + q'dt + q''\frac{dt^2}{2!} + \cdots\right),$$

where $a' = da/dt, a'' = d^2a/dt^2, \cdots$. The formula for the shortest distance between lines (1) and (2) is given by †

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*Koenigs: Géométrie réglée: *Annales de la Faculté des Sciences de Toulouse*, vol. 6, pp. 38–40, 61–63.

Joachimsthal: Anwendungen der Diff. und Int. Rechnung, etc., pp. 182–184.

Knoblauch: Einleitung in die allgemeine Theorie der krummen Flächen, pp. 104–106.

† Laurent: Traité d'analyse, vol. 2, pp. 298–303.

$$(3)\quad d=\frac{\begin{array}{r}\left(p'dt+p''\frac{dt^2}{2!}+\cdots\right)\left(b'dt+b''\frac{dt^2}{2!}+\cdots\right)\\ -\left(q'dt+q''\frac{dt^2}{2!}+\cdots\right)\left(a'dt+a''\frac{dt^2}{2!}+\cdots\right)\end{array}}{\sqrt{\left\{\left[b'dt+b''\frac{dt^2}{2!}+\cdots\right]^2+\left[a'dt+a''\frac{dt^2}{2!}+\cdots\right]^2\right.}\\ \left.+\left[a\left(b'dt+b''\frac{dt^2}{2!}+\cdots\right)-b\left(a'dt+a''\frac{dt^2}{2!}+\cdots\right)\right]^2\right\}}$$

and expanding, we get

$$(4)\quad d=\frac{\begin{array}{r}\left|\begin{smallmatrix}b'&a'\\q'&p'\end{smallmatrix}\right|dt^2+\left\{\frac{1}{2!\,1!}\left|\begin{smallmatrix}b''&a''\\q'&p'\end{smallmatrix}\right|+\frac{1}{1!\,2!}\left|\begin{smallmatrix}b'&a'\\q''&p''\end{smallmatrix}\right|\right\}dt^3+\left\{\frac{1}{3!\,1!}\left|\begin{smallmatrix}b'''&a'''\\q'&p'\end{smallmatrix}\right|\right.\\ +\frac{1}{2!\,2!}\left|\begin{smallmatrix}b''&a''\\q''&p''\end{smallmatrix}\right|+\left.\frac{1}{1!\,3!}\left|\begin{smallmatrix}b'&a'\\q'''&p'''\end{smallmatrix}\right|\right\}dt^4+\left\{\frac{1}{4!\,1!}\left|\begin{smallmatrix}b^{\mathrm{iv}}&a^{\mathrm{iv}}\\q'&p'\end{smallmatrix}\right|+\frac{1}{3!\,2!}\left|\begin{smallmatrix}b'''&a'''\\q''&p''\end{smallmatrix}\right|\right.\\ \left.+\frac{1}{2!\,3!}\left|\begin{smallmatrix}b''&a''\\q'''&p'''\end{smallmatrix}\right|+\frac{1}{1!\,4!}\left|\begin{smallmatrix}b'&a'\\q^{\mathrm{iv}}&p^{\mathrm{iv}}\end{smallmatrix}\right|\right\}dt^5+\cdots\end{array}}{\sqrt{\left[\left\{b'^2+a'^2+\left(ab'-ba'\right)^2\right\}dt^2+\left\{\frac{2}{1!\,2!}b'b''+\frac{2}{1!\,2!}a'a''\right.\right.}\\ +\frac{2}{1!\,2!}\left(ab'-ba'\right)\left(ab''-ba''\right)\Big\}dt^3+\left\{\frac{2}{1!\,3!}b'b'''\right.\\ +\frac{1}{2!\,2!}b''^2+\frac{2}{1!\,3!}a'a'''+\frac{1}{2!\,2!}a''^2+\frac{2}{1!\,3!}\left(ab'-ba'\right)\\ \left.\left.\times\left(ab'''-ba'''\right)+\frac{1}{2!\,2!}\left(ab''-ba''\right)^2\right\}dt^4+\cdots\right]}$$

or briefly,

$$(5)\quad d=\frac{N}{D}=\frac{[2]dt^2+[3]dt^3+[4]dt^4+[5]dt^5+\cdots}{\sqrt{\{[\bar{2}]dt^2+[\bar{3}]dt^3+[\bar{4}]dt^4+[\bar{5}]dt^5+\cdots\}}}.$$

It is at once evident that, in general, d is an infinitesimal of the first order; hence

THEOREM I. *The shortest distance between two consecutive generators of a skew surface, is, in general, an infinitesimal of the first order.*

If $[2]\equiv\left|\begin{smallmatrix}b'&a'\\q'&p'\end{smallmatrix}\right|=0$, $[3]$, which is $\frac{1}{2}d\,[2]/dt$, also vanishes, but $[4]$ does not vanish; then d becomes an infinitesimal of the third order. Now the condition that line (1) moves tangent to the space curve

$$(6)\qquad x=a\psi+p,\quad y=b\psi+q,\quad z=\psi$$

through every point of which one and only one of the lines (1) passes, is that

$$\psi=-\frac{p'}{a'}=-\frac{q'}{b'}\text{ or }\left|\begin{smallmatrix}b'&a'\\q'&p'\end{smallmatrix}\right|=0;$$

and curve (6) takes the form

$$(7)\qquad x=\frac{pa'-ap'}{a'},\quad y=\frac{qb'-bq'}{b'},\quad z=-\frac{p'}{a'},$$

where $b'p'=a'q'$.

Thus our line generates a developable surface. Hence

THEOREM II. *If the shortest distance between two consecutive positions of a moving line is an infinitesimal of the third order, the line will generate a developable surface, and conversely.*

Again, if $\left|\begin{smallmatrix} \nu & a' \\ q' & p' \end{smallmatrix}\right| = 0$ and $\left|\begin{smallmatrix} \nu'' & a'' \\ q'' & p'' \end{smallmatrix}\right| = 0$ simultaneously, we have, integrating these, the two sets of solutions (*i*) $b = ca + c_1$, $q = cp + c_2$ and (*ii*) $p = ka + k_1$, $q = kb + k_2$ for either of which every $\left|\begin{smallmatrix} b^{(n)} & a^{(n)} \\ q^{(n)} & p^{(n)} \end{smallmatrix}\right|$ vanishes identically, hence N vanishes identically, and d becomes zero. Now, under conditions (*i*) line (1) becomes

$$x = az + p, \quad y = (ca + c_1)z + cp + c_2 \tag{8}$$

and curve (7) takes the form

$$x = \frac{pa' - ap'}{a'}, \quad y = \frac{c(pa' - ap') + c_2a' - c_1p'}{a'}, \quad z = -\frac{p'}{a'}, \tag{9}$$

a curve lying in the plane $y = cx + c_1z + c_2$, hence the line moves tangent to a plane curve. Under conditions (*ii*) line (1) becomes

$$x = az + ka + k_1, \quad y = bz + kb + k_2 \tag{10}$$

and curve (7) takes the form

$$x = k_1, \quad y = k_2, \quad z = -k, \tag{11}$$

a point, and the line generates a cone. Hence

THEOREM III. *If the shortest distance between two consecutive positions of a moving line is identically zero, the line will move tangent to a plane curve or generate a cone, and conversely.*

We also note that $N = 0$ when $a' = 0$, $b' = 0$, or $a = \text{const.}$, $b = \text{const.}$, *i. e.*, our line moves parallel to itself and generates a cylinder; but this needs further discussion since D also vanishes here.

Finally, in any case, N is always an infinitesimal of even order.*

THEOREM IV. *For special lines of a surface (skew or developable), the shortest distance between two consecutive generators may be an infinitesimal of higher order than the third, but always of odd order.*

* Zindler: Liniengeometrie mit Anwendungen: Zweiter Teil, p. 13.

§ 2. *Discussion of the Denominator.*

The denominator of the expression for d is

$$(12)\quad D = \sqrt{\{[\bar{2}]dt^2 + [\bar{3}]dt^3 + [\bar{4}]dt^4 + \cdots\}}$$
$$= \sqrt{\{[b'^2 + a'^2 + (ab' - ba')^2]dt^2 + [b'b'' + a'a'' + (ab' - ba')(ab'' - ba'')]dt^3 + [\tfrac{1}{3}b'b''' + \tfrac{1}{4}b''^2 + \tfrac{1}{3}a'a''' + \tfrac{1}{4}a''^2 + \tfrac{1}{3}(ab' - ba')(ab''' - ba''') + \tfrac{1}{4}(ab'' - ba'')^2]dt^4 + \cdots\}}.$$

We have

$$[\bar{3}] = \tfrac{1}{4}\frac{d}{dt}[\bar{2}].$$

Hence

1) $[\bar{3}]$ will vanish identically whenever $[\bar{2}]$ vanishes identically, and

2) In general, D is an infinitesimal of the first order.

Let us find the solution of the differential equation

$$[\bar{2}] \equiv b'^2 + a'^2 + (ab' - ba,)^2 = 0.$$

The only real solutions of this equation are at once seen to be $b' = 0$, $a' = 0$, or $b = \text{const.}$, $a = \text{const.}$; these cause D to vanish identically. Hence

3) $b = \text{const.}$, $a = \text{const.}$ (the real solutions of $[\bar{2}] = 0$) cause D to vanish identically.

Now $[\bar{2}] = 0$ has imaginary solutions. To find these let us write the equation in the equivalent forms

$$\text{(i)}\quad (a^2+1)\left(\frac{db}{dt}\right)^2 + (b^2+1)\left(\frac{da}{dt}\right)^2 - 2ab\frac{db}{dt}\frac{da}{dt} = 0,$$

or

$$\text{(ii)}\quad b^2 - 2ab\frac{db}{da} + (a^2+1)\left(\frac{db}{da}\right)^2 + 1 = 0,$$

or

$$\text{(iii)}\quad b = a\frac{db}{da} \pm i\sqrt{\left(\frac{db}{da}\right)^2 + 1}.$$

This differential equation is in Clairaut's form, and its integral is

$$(13)\quad b = ca \pm i\sqrt{c^2+1}. \quad (c = \text{constant.})$$

This gives rise to the system of equations

$$b' = ca', \quad b'' = ca'', \quad b''' = ca''', \cdots, \quad b^{(n)} = ca^{(n)}, \cdots$$

and it is easily seen from (3) that D vanishes identically. Hence

4) The general solution $b = ca \pm i\sqrt{c^2 + 1}$ of the differential equation $[\bar{2}] = 0$ makes D vanish identically.

Now the differential equation $[\bar{2}] = 0$ is one of second degree and therefore has a singular solution. The latter is most readily found by solving the form (*ii*) as a quadratic equation in db/da, and setting the discriminant equal to zero. The singular solution is found to be

$$a^2 + b^2 + 1 = 0. \tag{14}$$

This solution (14) causes $[\bar{2}]$ and $[\bar{3}]$ to vanish and, as can easily be found with a little calculation, makes $[\bar{4}] = -b^4/4a^4$, *i. e.*, does not cause $[\bar{4}]$ to vanish. Hence

5) The singular solution $a^2 + b^2 + 1 = 0$ of the differential equation $[\bar{2}] = 0$ reduces D to an infinitesimal of the second order.

Again if $[\bar{4}]$ vanishes in addition to the vanishing of $[\bar{2}]$ through its singular solution, we must have, since $[\bar{4}] = -b^4/4a^4 = 0$, $b' = 0$, or $b = \text{const.}$, which taken in conjunction with $a^2 + b^2 + 1 = 0$, gives also $a = \text{const.}$; but $a = \text{const.}$, $b = \text{const.}$, cause D to vanish identically. Hence

6) The singular solution $a^2 + b^2 + 1 = 0$ of $[\bar{2}] = 0$ taken simultaneously with $[\bar{4}] = 0$ causes D to vanish identically.

Thus, we finally have

7) D is in general an infinitesimal of the first order; if $a^2 + b^2 + 1 = 0$, D is an infinitesimal of the second order; if $b = ca \pm i\sqrt{c^2 + 1}$, or if $b = \text{const.}$, a const., D is identically zero.

§ 3. *Geometric Interpretations.*

We have

$$d = \frac{N}{D} = \frac{[2]\,dt^2 + [3]\,dt^3 + [4]\,dt^4 + \cdots}{\sqrt{\{[\bar{2}]\,dt^2 + [\bar{3}]\,dt^3 + [\bar{4}]\,dt^4 + \cdots\}}}. \tag{15}$$

If $[\bar{2}] = 0$, through its general solution $b = ca \pm i\sqrt{c^2 + 1}$

and also $[2] \equiv \left|\begin{smallmatrix} b' & a' \\ q' & p' \end{smallmatrix}\right| = 0$, then $q'/p' = b'/a' = c$, or $q' = cp'$, and $q = cp + c_2$; but then both D and N vanish identically, and d takes the indeterminate form 0/0. But this indeterminacy is easily resolved, for the conditions $b = ca \pm i\sqrt{c^2+1}$, $q = cp + c_2$, are only a special case of $b = ca + c_1$, $q = cp + c_2$, the conditions that the line move tangent to a plane curve. Thus d is zero. The curve (6) becomes

$$(16)\quad x = \frac{pa' - ap'}{a'},\quad y = \frac{a(pa' - ap') + c_2 a' \mp i\sqrt{c^2+1}\,p'}{a'},\quad z = -\frac{p}{a'},$$

an imaginary curve lying in the imaginary plane

$$y = cx \pm i\sqrt{c^2+1}\,z + c_2.$$

THEOREM VI. *If a continuously moving straight line $x = az + p$, $y = bz + q$ obeys the conditions $b = ca \pm i\sqrt{c^2+1}$, $q = cp + c_2$, it will move tangent to an imaginary curve in an imaginary plane, the distance between two consecutive tangent lines being zero.*

Again, if $[\bar{2}]$ vanishes through its singular solution $a^2 + b^2 + 1 = 0$, then by 7) D is an infinitesimal of the second order, and if $[2]$ does not vanish, N is also an infinitesimal of the second order, and thus d is finite. Now the relation $a^2 + b^2 + 1 = 0$ expresses that the sum of the squares of three quantities proportional to the direction cosines of our line, is equal to zero, which property is the distinguishing characteristic of a minimal straight line of space. Hence

THEOREM VII. *The shortest distance between two consecutive generators of a skew surface generated by a continuously moving minimal straight line, is finite.*

If in addition to the vanishing of $[\bar{2}]$ through its singular solution $a^2 + b^2 + 1 = 0$, $[2] = \left|\begin{smallmatrix} b' & a' \\ q' & p' \end{smallmatrix}\right|$ also vanishes, N is an infinitesimal of the fourth order, for we have

$$a^2 + b^2 + 1 = 0 \quad \text{and} \quad \begin{vmatrix} b' & a' \\ q' & p' \end{vmatrix} = 0$$

or

$$aa' + bb' = 0 \quad \text{and} \quad \frac{b'}{a'} = \frac{q'}{p'} = -\frac{a}{b}.$$

$$\therefore q' = -\frac{a}{b}p' \quad \text{or} \quad q = -\int \frac{a}{b}p'\,dt\,.$$

$$\therefore b = \pm i\sqrt{a^2+1} \quad \text{and} \quad q = \pm i\int \frac{a}{\sqrt{a^2+1}}p'dt.$$

With these conditions we can easily calculate

$$\begin{vmatrix} b'' & a'' \\ q'' & p'' \end{vmatrix} = \frac{a'}{b^3}(a''p' - a'p'') \neq 0,$$

hence [4] does not vanish, *i. e.*, N is an infinitesimal of the fourth order. Thus d is an infinitesimal of the second order. The line moves tangent to a minimal curve in space. The equations of the line are

$$(17) \qquad x = az + p, \quad y = \pm i\sqrt{a^2+1}\,z \pm i\int \frac{a}{\sqrt{a^2+1}}p'dt,$$

and those of the minimal curve to which the line moves tangent are, from (6),

$$(18) \qquad x = \frac{pa' - ap'}{a'}, \qquad z = -\frac{p'}{a'},$$

$$y = \mp i\frac{\sqrt{a^2+1}\,p'}{a'} \pm i\int \frac{a}{\sqrt{a^2+1}}p'dt.$$

Hence

THEOREM VIII. *The shortest distance between any two consecutive generators of the tangential surface to a minimal curve in space, i. e., of a minimal developable, is an infinitesimal of the second order; and conversely, if a line moves so that the shortest distance between any two consecutive positions is an infinitesimal of the second order, it will generate a minimal developable.*

The converse as stated in Theorem VIII is easily deduced from the above discussion, for d is an infinitesimal of second order only if $[2] = 0$, $[4] \neq 0$, $[\bar{2}] = 0$ (through its singular solution) and $[\bar{4}] \neq 0$.

From equations (18) we have

$$x' = \frac{a(p'a'' - a'p'')}{a'^2}, \quad y' = \frac{\pm i\sqrt{a^2+1}(p'a'' - a'p'')}{a'^2},$$

$$z' = \frac{p'a'' - a'p''}{a'^2}$$

and hence, the equations of the minimal curve may be written

$$(19)\quad x=\int\frac{a(p'a''-a'p'')}{a'^2}dt,\quad y=\pm i\int\frac{\sqrt{a^2+1}(p'a''-a'p'')}{a'^2}dt,$$

$$z=\int\frac{p'a''-a'p''}{a'^2}dt.$$

Now set $a=(1-\tau^2)/2\tau$, where τ is an arbitrary function of t,

$$\therefore\pm i\sqrt{a^2+1}=\frac{i(1+\tau^2)}{2\tau},$$

and set $(p'a''-a'p'')/a'^2dt=\tau F(\tau)d\tau$, where $F(\tau)$ is an arbitrary function of τ. Then equations (19) become

$$(20)\quad x=\frac{1}{2}\int(1-\tau^2)F(\tau)d\tau,\quad y=\frac{i}{2}\int(1+\tau^2)F(\tau)d\tau,$$

$$z=\int\tau F(\tau)d\tau,$$

the well-known equations of the minimal curve.

Using equations (20), we have

$$[2]=0,\ [3]=0,\ [\bar{2}]=0,\ [3]=0,\ [4]=\frac{-iF(\tau)}{12\tau^2},\ [\bar{4}]=-\frac{1}{4\tau^4}$$

and

$$(21)\quad d=\frac{\dfrac{-iF(\tau)}{12\tau^2}d\tau^4+\cdots}{\sqrt{\left\{-\dfrac{1}{4\tau^4}d\tau^4+\cdots\right\}}}=-\frac{1}{6}F(\tau)d\tau^2+\cdots$$

Using equations (19), we have $[4]=-a'/12b^3\times(p'a''-a'p'')$, $[\bar{4}]=-b'^4/4a^4$ and

$$(22)\quad d=\frac{i}{6}\frac{p'a''-a'p''}{ba'}dt^2+\cdots=\frac{p'a''-a'p''}{6a'\sqrt{a^2+1}}dt^2+\cdots.$$

Finally we may have $[\bar{2}]=0$ through its singular solution $a^2+b^2+1=0$ and $[4]=0$, *i. e.*, $a=$ const., $b=$ const. Here both N and D are zero, and hence d has the indeterminate form $0/0$. Since $\psi=-p'/a'=-p'/0=\infty$, the curve (6) becomes $x=\infty, y=\infty, z=\infty$, *i. e.*, our lines all pass through the same point at infinity; but for the purpose in hand we cannot con-

sider the cylinder as a special case of the cone, *i. e.*, a cone whose vertex is at infinity, for when $a = \text{const.}$, $b = \text{const.}$, the conditions that the line generates a cone, viz., $p = ka + k_1$, $q = kb + k_2$ become $p = \text{const.}$, $q = \text{const.}$ and the two consecutive generators actually coincide. Thus, we cannot say, that the shortest distance between two consecutive generators of a cylinder is zero, *i. e.*, that the two generators actually intersect. There is no shortest distance between two such lines; they are everywhere equally distant. Hence, to find the distance between two consecutive parallel lines, we shall have to use the formula for the distance of a point from a line. It is easily seen that, in general, this distance is an infinitesimal of the first order; it is zero only if the two consecutive lines coincide; it is infinite when $a^2 + b^2 + 1 = 0$. Hence

THEOREM IX. *The distance between two consecutive generators of a cylinder is, in general, an infinitesimal of the first order; if the generator is a minimal straight line, the distance is infinite.*

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NOTE ON THE COMMUTATOR OF TWO OPERATORS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 27, 1907.)

THERE is a confusing lack of uniformity with respect to the use of the term commutator. The present note aims to exhibit this fact and to point out some of its sources in the hope that these data may tend towards greater uniformity in the use of this term and also make its various meanings less confusing to the reader.

The operation now known as the commutator of two operators was used for a long time in the development of group theory before it received a special name. It is frequently employed, in various forms, in Jordan's Traité des substitutions, and its elegant application in the study of direct products was recognized by Hölder* and others. The first paper which deals with the

*Hölder, *Math. Annalen*, vol. 34 (1889), p. 35. It should be noted that the reference 91) in Encyklopädie der mathematischen Wissenschaften, vol. 1, p. 219, should be to this article instead of to the later one in vol. 43.

important properties of the subgroups formed by the commutators of every pair of operators in a finite group appeared in the *Quarterly Journal of Mathematics* in 1896.* In this paper the commutator of s and t is represented by $sts^{-1}t^{-1}$, but no special name is assigned to the operation. Its form was suggested by the form of the alternant (Klammerausdruck) in the theory of continuous groups.

Towards the close of 1896 Frobenius reproduced some of the fundamental theorems relating to the commutator subgroup † (giving due credit both to the earlier publication and to Dedekind's unpublished work along the same line) and called the operation $s^{-1}t^{-1}st$ the commutator of t and s, following Dedekind.

Shortly after this Dedekind published an article in the *Mathematische Annalen*‡ in which he gave some of the known properties of the commutator subgroup and defined $s^{-1}t^{-1}st$ as the commutator of t and s just as Frobenius had done. About a year later $s^{-1}t^{-1}st$ was defined § as the commutator of s and t, instead of the commutator of t and s as had been done in the articles just cited, and it was observed that the interchange of s and t in $s^{-1}t^{-1}st$ gives its inverse, thus leading to two commutators of s and t if the order of these operators is not observed.

In Weber's Lehrbuch der algebra, volume 2 (1899), page 133, the second definition of commutator given above is adopted, so that Weber's commutator is the inverse of that given by Frobenius and Dedekind. Some writers say that both of the two operators $s^{-1}t^{-1}st$ and $t^{-1}s^{-1}ts$ are commutators of s and t, thus giving a double meaning to this term. This definition appears implicitly in the last article cited above and is adopted by Easton in his Constructive development of group theory, 1902, page 57 and also by De Séguier in his Groupes abstraits, 1904, page 8. A disadvantage of this definition is that the expression "commutator of s and t" may mean either one of two operators.

If the elements of the commutator $s^{-1}t^{-1}st$ are permuted in every possible manner there result eight operators which may be distinct and differ from the identity. Each of them has the

* Vol. 28, p. 266.
† Frobenius, *Berliner Sitzungsberichte*, 1896, p. 1348.
‡ Dedekind, *Math. Annalen*, vol. 48 (1897), p. 553.
§ BULLETIN, vol. 4 (1898), p. 135.

property that it is explicitly the transform of an operator multiplied by its inverse, and hence all of them have been called commutators.* Four of them are conjugate under $\{s, t\}$, while the others are the inverses of these four. Hence they have the same order and all of them occur in the commutator subgroup of the group generated by s and t. In fact they generate this subgroup. As long as we are concerned only with the properties of the commutator subgroup, it does not matter which of the eight forms considered above is regarded as the commutator of s and t. In fact, all of them may be regarded as commutators of these elements in this connection. In view of the fact that the commutator subgroup, has played the principal rôle in the discussions in which the term commutator has been employed, it is not surprising that there should have been some laxity in the definition of the term.

While the eight commutators considered above play the same rôle with respect to the commutator subgroup, they have quite different properties if considered as factors of s and t. For instance, $s^{-1}t^{-1}st$ is the factor which multiplied on the right into s $(s \cdot s^{-1}t^{-1}st)$ gives the transform of s with respect to t, but it has no such property as regards t. In fact, the product of this commutator into t will generally have an order which differs from the order of t and hence there is, in general, no operator which transforms t into itself multiplied by this commutator. From this it follows that $s^{-1}t^{-1}st$ has entirely different properties with respect to the two operators s, t and the question arises whether it would not be desirable to select for the commutator of s, t an operator whose properties with respect to s and t are more nearly alike.

Two of the eight commutators ($s^{-1}t^{-1}st$, $t^{-1}sts^{-1}$, $sts^{-1}t^{-1}$, $ts^{-1}t^{-1}s$, $t^{-1}s^{-1}ts$, $st^{-1}s^{-1}t$, $tst^{-1}s^{-1}$, $s^{-1}tst^{-1}$) which are obtained by permuting the elements of $s^{-1}t^{-1}st$ are such that the products obtained by multiplying them into s or t will be of the same order as s or t respectively. Two others will not change the order of s, but will generally change the order of t, if used

*BULLETIN, vol. 5 (1899), p. 239. The general definition of a commutator is "the product of the transform of an operator and its inverse." On this point all authors agree. The disagreements relate to the definition of a commutator by means of its elements. When we speak of the commutators of a group it is assumed that the elements of the commutators are also found in the group and hence it may happen that only a small number of its operators are commutators. Since every operator of a group may be represented by a positive substitution and all positive substitutions are commutators, it follows that every possible operator is a commutator of some elements.

as factors with s or t, while two others will not change the order of t but will change the order of s. The remaining two will, in general, change the order of both t and s if they are multiplied into these operators respectively. From this standpoint it does not appear desirable to call either $ts^{-1}t^{-1}s$ or $st^{-1}s^{-1}t$ a commutator of s and t. That is, if we regard the commutator as a factor which must be multiplied into an operator in order to obtain its conjugate (and this is a very useful concept) neither of the two operators $ts^{-1}t^{-1}s$, $st^{-1}s^{-1}t$ is a commutator of s or t since we cannot generally transform s or t into themselves multiplied by one of these two operators. From this point of view $s^{-1}t^{-1}st$ and $tst^{-1}s^{-1}$ are commutators of s but not of t, $t^{-1}s^{-1}ts$ and $sts^{-1}t^{-1}$ are commutators of t but not of s, while $t^{-1}sts^{-1}$ and $s^{-1}tst^{-1}$ are commutators of both s and t.

From the preceding paragraph it follows that for some important uses of the commutator it would appear desirable to call $t^{-1}sts^{-1}$ the commutator of s and t, and $s^{-1}tst^{-1}$ the commutator of t and s, while $s^{-1}t^{-1}st$ and $t^{-1}s^{-1}ts$ might be called the commutators of s with respect to t and of t with respect to s respectively. Although the main objects of this note are to call attention to the history and the nature of the commutator, yet the reasons given above seem to demand a new definition of the term and we shall hereafter call $t^{-1}sts^{-1}$ *the commutator of* s *and* t. If this is done the eight commutators mentioned at the beginning of the preceding paragraph are respectively the commutators of t^{-1} and s, s and t, t and s^{-1}, s^{-1} and t^{-1}, s^{-1} and t, t^{-1} and s^{-1}, s and t^{-1}, t and s. These eight commutators are conjugate under the octic group.

It is well known that every simple isomorphism of a group G with itself may be obtained by transforming all of its operators by some operator in its holomorph. In such an isomorphism each operator corresponds to itself multiplied by some operator of G. If G involves s and t, and if we transform G by t, the operator which corresponds to s may be obtained by multiplying s on the left by the commutator of s and t. By multiplying t on the right by the same commutator we obtain its transform with respect to s^{-1}. As in many instances (such as finding the operators of G which transform a function into one of its conjugates) it is desirable to employ left hand multiplication, the advantages of the last definition of commutator become apparent. In conclusion it may be said that it seems very desirable that the term commutator should

be completely defined by giving its elements in order. That is, the expression 'commutator of s and t' should not have a double meaning. For the most important applications which have been made of commutators any one of the given definitions seems just as good as any other, but there are applications in which the last definition seems to be the most convenient. It may be added that the definition of commutator in the Encyklopädie der Mathematischen Wissenschaften, Volume I 1, page 210, is rendered meaningless by typographical errors.

A THEOREM IN THE THEORY OF NUMBERS.

BY PROFESSOR D. N. LEHMER.

(Read before the San Francisco Section of the American Mathematical Society, December 19, 1903.)

LAGRANGE has shown that if the indeterminate equation $x^2 - Ry^2 = \pm D$ is resolvable in integers, D being less than $\sqrt{R}$, and x and y being relative primes, then D is a denominator of a complete quotient in the expansion of $\sqrt{R}$ in a continued fraction. (For a proof of this theorem, see Chrystal's Algebra II, page 451.) Making use of this result, we may prove the following interesting theorem, which is sometimes very effective in finding the factors of large numbers.

If R is the product of two factors which differ by less than $2\sqrt[4]{R}$, these two factors may be found directly from the expansion of $\sqrt{R}$ in a continued fraction.

Let the two factors be p and q, so that $R = pq$. Then $R = [\frac{1}{2}(p+q)]^2 - [\frac{1}{2}(p-q)]^2$, and the equation $x^2 - Ry^2 = [\frac{1}{2}(p-q)]^2$ is resolvable in integers. If now $[\frac{1}{2}(p-q)]^2$ is less than $\sqrt{R}$, then by the theorem quoted above, there will be a denominator of a complete quotient in the expansion of $\sqrt{R}$ equal to $[\frac{1}{2}(p-q)]^2$. Since $[\frac{1}{2}(p-q)]^2 < \sqrt{R}$, then $p - q < 2\sqrt[4]{R}$. Moreover the values of the indeterminates in the equation $x^2 - Ry^2 = \pm D$, are furnished by the numerator and denominator of the convergent which immediately precedes the complete quotient having D for a denominator. Hence it follows that the expansion of $\sqrt{R}$ need not be carried farther than is sufficient to make the numerator of the convergent as

great as R. The method may be applied to κR, and we have the following theorem:

THEOREM: *If no denominator of odd rank, after the first, in the complete quotients obtained by expanding $\sqrt{\kappa R}$ in a continued fraction, turns out to be a perfect square, the expansion being carried out until the numerator of the last convergent is greater than κR, then the factors of κR differ by more than $2\sqrt[4]{\kappa R}$.*

As an example of the application of this theorem to the discovery of prime factors, take Jevons's * number $8616460799 = R$. No perfect square appears in the denominators of the complete quotients obtained in expanding $\sqrt{R}$, whence one infers that the factors differ by more than $608 = 2\sqrt[4]{R}$. Similar failure attends the expansion of $\sqrt{2R}$, $\sqrt{6R}$, and $\sqrt{30R}$. On expanding $\sqrt{210R}$, however, the third denominator is found to be the square $11881 = 109^2$. The numerator of the second convergent is 2690321; we know then that the numbers 2690321 ± 109 contain the desired factors of the given number. The factors are 89681 and 96079.

The most advantageous value of κ to take is the product of the smallest distinct primes. Thus if $\kappa = 30$ and the factors of R are p and q, then the factors will be discovered if $p - 30q$, $2p - 15q$, $3p - 10q$, or $5p - 6q$ are less than $2\sqrt[4]{30R}$.

BERKELEY, CAL.

PROJECTIONS OF THE GLOBE APPROPRIATE FOR LABORATORY METHODS OF STUDYING THE GENERAL CIRCULATION OF THE ATMOSPHERE.

BY PROFESSOR CLEVELAND ABBE.

THE general circulation of the atmosphere is controlled by the general distribution of land and water, and by the insolation, with its resultant temperature, evaporation and clouds. In the analytic treatment of this problem, beginning with D'Alembert, Ferrel, and Erman, as well as in the more elegant works

* Jevons, Principles of science, p. 123, "Can the reader say what two numbers multiplied together will produce the number 8616460799? I think it unlikely that anyone but myself will ever know." I think that the number has been resolved before, but I do not know by whom.

of Helmholtz, Oberbeck and Margules, it has always been considered necessary to simplify the problem by assuming a uniform surface and uniform coefficient of resistance for the whole globe, as also a uniform condition of dry air without the complexity produced by the presence of clouds. In this shape the problem is already too difficult for pure analysis, and writers have generally achieved the special solutions only by introducing numerical values based upon observations of temperature, pressure, and winds that represent existing conditions upon the earth's surface. It is likely that the profound meteorological problem, namely the *deduction* of the actual winds that necessarily result from the laws of mechanics, as applied to the earth, will for a long time be too difficult for pure analysis; but the solution of this problem is greatly to be desired, and on many occasions I have stated my belief that it may be arrived at in an experimental way provided we have a laboratory properly equipped for meteorological investigations. We cannot reproduce atmospheric conditions surrounding a small globe in the laboratory, because in the free atmosphere the force of gravity is directed normal to the earth's surface, whereas in the laboratory it would be inclined to a globe at every possible angle. We must, therefore, represent any portion of the globe, with its atmosphere, by a horizontal plane surface covered with some heavy gas or liquid. The most natural thought is to adopt polar projections of the northern and southern hemispheres respectively, bounded at or beyond their equators by flexible rims that shall represent the assumed irregular variable equatorial boundary between the northern and southern circulations. This will do for those months of the year during which we may assume that the northern and southern systems do not intrude upon each other to any great extent; but of course this will fail entirely when the great Asiatic monsoon is in progress and for that season a very different laboratory experiment must be devised. We therefore for the present confine ourselves to the months October to March. Our projections of the two hemispheres, being covered with a thin layer of carbonic acid gas (or for that matter any liquid such as alcohol or water can be used as well), are to be set in steady rotation, and are to be warmed from below in such a way as approximately to imitate the actual isotherms of the lower atmosphere for any moment or day. Not only the general distribution of temperature in latitude, but the special areas of high and low temperatures, can easily be imitated by elec-

trical resistance coils. In order that the thin layer which imitates the atmosphere shall be free from the action of the surrounding air of the laboratory our rotating planospheres are to be covered by plates of glass rotating with them, so that the upper layer of air in the model shall, like the upper layer in the atmosphere, rotate with the surrounding ether.

We have thus imagined a construction that has a certain similarity but many points of dissimilarity to the atmosphere of the earth. Our problem is to reduce these latter to a minimum, and then so to interpret the observed motions in our model that we may infer something as to the motions in the earth's atmosphere. To do this our first step must be to consider the ideas laid down by Helmholtz as to the "mechanical similarity" of the atmosphere of the earth and the model atmosphere. Various theorems relative to this similarity were first published by Helmholtz in 1873, "On a theorem relative to movements that are geometrically similar in fluid bodies," and these ideas were further applied by him in later memoirs on atmospheric motions, in 1888 and 1889. He shows how the movements in a small model may be made geometrically and mechanically similar to those of the earth's atmosphere. The equations of motion for the model atmosphere become analogous to those of the free atmosphere by introducing certain conditions represented by three constant factors, *i. e.*, I, the viscosity of the model or liquid is to be a certain multiple q of the viscosity of the atmosphere; II, the density in the model is to be a certain multiple r of the density of the atmosphere; III, the velocities along three coordinate axes in the model are to be equal to those in the atmosphere multiplied by a certain constant factor n. Under these conditions the pressures in the model will be n^2r times the pressures in the atmosphere. The linear distances in the model will be q/n times the linear distances in the atmosphere, and the time argument in the model will be q/n^2 times the time element in the free air. It is assumed that the relative movements are not so violent as to depart greatly from what is known as laminar or lamellar motion, and that the viscosity may be combined with the resistances due to convection and to the formation of discontinuous motions. Numerous applications of these principles are given by Helmholtz, and some further developments have lately been added by Lord Rayleigh.

But in applying these ideas to our two polar projections we

stumble at once upon a great difficulty, namely that the maps are not true representations of the spherical surface of the earth. Every projection on a plane must distort the geometric relations between the points upon the original spherical surface, and students must always choose a style of projection that suits the problem they have on hand. In the present case (1) we should like to preserve the equality of surface areas, since that is proportional to the mass of the air above it; (2) we should like to preserve equality of distances, since we have to measure and compare velocities; but, above all, from a mechanical point of view, (3) we should like to preserve the equality of the moments of inertia, that is to say the moment of inertia of a unit mass relative to the earth's axis should be a constant fraction of the moment of inertia of the corresponding unit mass in the model with reference to the pole about which it is rotating. These three desiderata would respectively necessitate the use of (1) an equal surface development; (2) a polyconic development; and (3) an orthographic projection. There are still other conditions with projections that correspond thereto. That which best satisfies all desiderata with the minimum sum total of errors seems to be Airy's "projection by balance of errors," published by him in 1861 as suggested by the projection used by Sir Henry James, in which latter the point of view was at the distance from the surface of the sphere equal to the radius. James's idea was to obtain a projection that should include a very large part of the surface of the earth at one view; and the map that he produced did in fact include a radius of 120° 0′; this was a projection, properly so called. Airy, however, solved the problem for a development, properly so called, which should respond to the condition that the square of the distortion in distance plus the square of the distortion in area should be a minimum when one considers the sum total of the whole surface. This minimum would necessarily depend on the total area included in the map. A few months later R. A. Clarke showed that there was a slight error in Airy's computations, and then worked out a slight modification of the problem. Clarke deduced a projection, properly so called, allowing both the visual point and the plane of projection to be adjustable, so as to satisfy the same condition as that adopted by Airy in his development. The formulas and computations by Clarke's projection are far more troublesome than for Airy's development, especially when the latter are arranged in the way that I have

given. The only advantage that Clarke's projection can have is that it can be constructed graphically without the use of numerical tables or computations; but this advantage disappears as soon as we attempt to construct a large map.

The combination of Airy's development (or some projections still more appropriate) with Helmholtz's method of mechanical similarity should enable us to interpret our laboratory experiments intelligently, so that from these we may construct a close approximation to the general circulation of the atmosphere.

I consider it extremely desirable that these experiments should be made on a large scale, with due regard to all numerical, statistical and mechanical details in some laboratory where the study of meteorology is prosecuted as a branch of mathematical physics.

SHORTER NOTICES.

Breve Storia della Matematica dai tempi antichi al medio evo. By GAETANO FAZZARI. Milano, R. Sandron, 1907. 268 pp. Price, 4 lire.

IT is rather strange that Italy, the country that produced the most learned bibliophile in the domain of mathematics, Prince Boncompagni, and that furnished to France another well-known collector of early works, the historian Libri, should have published so little relating to the general development of the science. It is true that Favaro and Loria have contributed very acceptably to the history of certain periods or topics, and that Riccardi's bibliography of the early mathematical works of his own country will always be a standard book of reference, but in spite of all the encouragement of men like these, and all the patriotism that would lead an Italian to write the story of a science that so largely developed on his native soil, such a work as a worthy general history of mathematics does not exist in the language of Italy. It is for this reason that such an attempt as Professor Fazzari's should be particularly welcome, the more so as it was written on the island in which Archimedes spent most of his life, which Pythagoras visited, and to which Maurolycus brought no small amount of glory in the period of the Renaissance.

The aim of Professor Fazzari has been to write a work of about the character of Ball's and Cajori's popular histories, one

that tells the story of mathematics in such a simple manner as to appeal to the younger student rather than to the mature scholar, leading him to higher fields in the science by an interesting recital of its progress. He has not, therefore, felt it necessary to pay attention to the original sources of information, but has gathered his material from a few of the standard writers, setting it forth in a pleasing conversational style. The general range of the subjects may be seen from the table of contents: Chapter I, Decimal numeration; Chapter II, The Egyptians; Chapter III, The Babylonians; Chapter IV, Logistica among the Greeks; Chapter V, The pre-euclidean period, including Thales and the Ionic school, the mathematics of the fifth century B. C., and the Academy; Chapter VI, The golden period of Greek geometry; Chapter VII, The Greek mathematics of the second century B. C.; Chapter VIII, The period of decadence; Chapter IX, The Romans; Chapter X, The Indians; Chapter XI, The Arabs; Chapter XII, The Byzantine school; Chapter XIII, The middle ages, from the seventh to the fifteenth centuries inclusive. It will thus be seen that the general field covered is about what would be expected, and that the result must be helpful to students and teachers in Italy if the work has been tolerably well done.

It is a little difficult to answer the question which the preceding sentence suggests. What is meant by "tolerable" in such a work? Where does the intolerable begin, and what is the norm of comparison? The book is better than Hoefer's, as in the natural order of things it could hardly help being; it is not so good as Ball's, and indeed it would be difficult to improve upon the popular style of that writer; it is not in the same class with such scholarly productions as those of Montucla, Libri, and Cantor, and of course this was not to be expected. It is but just to say that it is pleasantly written and that it covers the leading topics down to the opening of the sixteenth century; but it cannot be said that it shows a very wide range of reading, or that Professor Fazzari has produced an entirely reliable work.

With respect to the authorities consulted, the care shown in securing data, and the weighing of evidence, a few illustrations will suffice. In speaking of the Roman numerals, but two theories are given to explain the symbols, both rather antiquated, and no evidence is shown of any knowledge of the important investigations of Friedlein, Hoüel, Zangemeister, Mommsen,

and R. Bombelli. As a consequence, the treatment of the subject has no real value, if, indeed, it can be called fairly reliable. In the rather fanciful etymologies of the lower numerals, the Humboldt-Curtius theory as to *quinque*, now generally agreed to be untenable, is given as the latest pronouncement in the case. The illustration (Fig. 3) from the Ahmes Papyrus is not only incorrectly drawn (having the hieroglyphic for 37 instead of 33), but it is not from the papyrus at all, being merely Eisenlohr's hieroglyphic rendering of the hieratic original. The positive statement that Heron of Alexandria was living in the period 120–100 B. C. shows that the author has not consulted Schmidt's 1899 edition of the Opera, and the passing of the name of Menelaus with no reference to the anharmonic ratio shows that he has not consulted the Abhandlungen for 1902. The treatment of algebraic symbolism (page 127) leaves an entirely wrong impression as to the early use of the common signs of operation and relation, and it seems doubtful if the author himself is entirely clear in this matter. On the question of the origin of the Hindu-Arabic numerals evidently no attempt has been made to trace these forms back to any of the pre-Christian cave inscriptions, and the statement on page 177 indicates that the author is not aware of the existence of these sources. That the numerals came primarily from the Sanskrit alphabet, as stated on page 179, is now an abandoned theory.

These criticisms fairly represent the lack of scholarly effort in the preparation of the work. Similar ones will suggest themselves to the reader at intervals so frequent as to lead him to regret that the author did not put more time upon the preparation of his manuscript and revise the proof more carefully.

As to the description of the Babylonian numerals, it was not to be expected that the recent Hilprecht discoveries would be known to Professor Fazzari, but these make his treatment of the matter seem very elementary.*

As a piece of bookmaking the work leaves much to be desired, and the absence of an index in such a publication is indefensible. Of the minor errors, chiefly in the proof-reading, the following are types: Händbuch for Handbuch (p. 19), the letter sigma for stigma (p. 35), und for and (p. 56), 1491 for 1494 (Paciuolo, p. 127), Ceilan for Ceulen (p. 171), Geometria for La géométrie as the title of the first edition of Descartes (page 184), and the double spelling of Bhaskara (pages 153, 173)

*See the BULLETIN, vol. 13, p. 392.

and Paciuolo (pages 127, 259). The form "Mohammed ben Musa Al Hovarezmi" is probably the least satisfactory of any for the name of the great Arab mathematician, particularly as it is followed by the statement that he was a native of Chwarizm, and as the form "Alhowarizmi" appears on page 207. It is unfortunate that we have as yet no generally accepted norm for such transliterations, but there is no good authority for such a mixture of languages as this. A similar criticism might justly be passed upon most of the other oriental names in the work, particularly Al Fahri (page 194), Al Karhi and Alkarhi (pages 194, 195), and Alhayyami (page 199).

DAVID EUGENE SMITH.

Leçons de Géométrie Supérieure. Professées en 1905–1906 par M. E. VESSIOT. Lyon, Delaroche et Schneider, 1906. 4to., 326 pp. (autographed).

THESE lectures delivered by Vessiot during the year 1905–1906 were published in the present form at the demand of his students. The author remarks in the preface that he is hopeful that they may be of service to those who are beginning the study of higher geometry and may serve them as a good preparation for the reading of original memoirs and such works as Darboux's Théorie des surfaces. It is the opinion of the reviewer that the lectures serve these purposes admirably. The attack is direct and the end to be reached is kept clearly before the reader, in fact the whole presentation is such as to lead the beginner to an appreciation of the subject. A glance at the table of contents will convince one that the book will serve as a good introduction to the study of Darboux.

The principal object of the lessons is the study of systems of straight lines but owing to the close relation between lines and spheres it is quite natural that systems of spheres should be studied also. It is assumed at the outset that the student is familiar with the elementary notions of twisted curves and surfaces (tangent planes, tangent lines, etc.), and that he has some acquaintance with the elements of the theory of contact.

In Chapter I, Frenet's formulas for twisted curves are derived and the simple properties of developable surfaces obtained. The rectifying and polar surfaces are discussed as examples of developables. Chapters II, III, and IV are devoted to the general surface theory. Throughout these chapters the importance of the two differential forms of Gauss

$$\Phi(du, dv) = Edu^2 + 2Fdu\,dv + Gdv^2,$$
$$\Psi(du, dv) = E'du^2 + 2F'du\,dv + G'dv^2$$

is insisted upon quite emphatically. These chapters are only introductory to the discussion of systems of lines and spheres, so quite naturally conjugate directions, asymptotic lines, geodesic lines and lines of curvature are the principal subjects treated. In these chapters good geometric interpretations are given to nearly all the analytic results. In Chapter V the foregoing theory is applied to scrolls and developable surfaces and the general properties of these surfaces are derived. Chapters VI, VII, and VIII are devoted to the study of congruences of lines and the correspondence set up by such congruences. Chapter VII is given up entirely to normal congruences and their applications. In this chapter the close relation between lines and spheres is first pointed out and discussed in an elementary manner. The analogy between asymptotic lines and lines of curvature is also pointed out. In Chapter VIII homogeneous and tangential coordinates are introduced. Here the lines of the congruence are defined by corresponding points on two surfaces between which a correspondence has been established. The dual of any line of the congruence is the line of intersection of the tangent planes at the points which define the line of the congruence. Chapters IX and X treat the general and linear line complex; curves and surfaces belonging to a congruence are the principal subjects discussed. Chapter IX is devoted to dualistic transformations and the transformation of Lie. A short discussion of contact transformation is given and then applied to dualistic transformations and the line-sphere transformation of Lie. In this chapter the correspondence between lines and spheres is again taken up and discussed more in detail by means of the Lie transformation.

Chapter XII deals with triply orthogonal systems. After demonstrating Dupin's theorem: On each surface of a triply orthogonal system the intersections with the other surfaces of this system are lines of curvature, — it is applied to the discussion of triply orthogonal systems which contain given surfaces as part of the system. Chapter XIII discusses congruences of spheres and cyclical systems. In this the concluding chapter spheres are discussed without reference to Lie's transformation. At the beginning of the discussion of focal points of a congruence of spheres another theorem of Dupin is proved: A normal

congruence of lines is reflected or refracted on any surface into a normal congruence. It is shown that each sphere of the congruence touches the focal surface in two points. The congruence of lines formed by joining these points is discussed and some very pretty relations between the surface of centers of the spheres and their envelope are derived therefrom. The chapter closes with a short account of the cyclical systems of Ribaucour and Weingarten surfaces.

An excellent set of exercises is given to accompany each chapter. C. L. E. MOORE.

Nichteuklidische Geometrie. Von HEINRICH LIEBMANN. Leipzig, G. J. Göschen (Sammlung Schubert, XLIX), 1905. 12mo. viii + 248 pp.

IN this volume of the Schubert collection, Professor Liebmann has succeeded in presenting an introduction to non-euclidean geometry that is brief, readable, and well-balanced. Its brevity will recommend it to the student whose interest in the subject has been aroused by the numerous references in literature, but whose time and maturity are scarcely sufficient for a study of the many longer and more difficult works. It might well appeal also to a teacher of elementary geometry. The recent literature on non-euclidean geometry naturally falls into two classes: the one dealing with the lives and writings of Lobachevsky, Bolyai, and Gauss; and the other consisting of systematic developments of particular phases of the subject. In this book there is a happy combination of the two methods, giving a broad outlook, and yet not sacrificing the unity.

The first chapter contains an interesting account of the parallel axiom and of the attempts at its proof, considered from an historical point of view. The next five chapters, comprising three fourths of the book by pages, are devoted to hyperbolic geometry, beginning with a very simple account of its picturing by means of circles in the euclidean plane. One regrets that references are not given here to some, at least, of the articles that have appeared during the last twenty years on this picturing. With this one exception, the many exact references to a comparatively wide range of literature form one of the most attractive features of the book. There are other chapters on hyperbolic geometry, dealing with the synthetic and the analytic geometry and the trigonometry in the hyperbolic plane. After

this more detailed account, the spherical-elliptical geometry is disposed of in one short chapter. The last chapter is on non-euclidean mechanics; and the book is concluded by a few pages in which attention is directed to some of the discussions about the nature of actual space.

E. B. COWLEY.

A Brief Introduction to the Infinitesimal Calculus. Designed especially to aid in reading mathematical economics and statistics. By IRVING FISHER, Ph.D., Professor of political economy in Yale University. Second edition. New York and London, The Macmillan Company, 1906. 12mo. xiii + 84 pp. Price, 75 cents.

THIS book gives an excellent bird's-eye view of the differential calculus, and indeed of the integral calculus. It is written with remarkable clearness, the illustrations from geometry, physics and economics being well chosen and well placed. In this, the second edition, the notion of the "little zero" is not used. Its use in the first edition was criticized by Professor Fiske in his review of the book in the BULLETIN, February, 1898, page 238.

Though small, the book is very comprehensive. If it were to be enlarged, the first addition would perhaps be an article on the mean value theorem, of which article 69 is suggestive, and a page or two on integration as summation, in place of the two short articles 76, 87. Some footnotes, such as the one inserted in the German edition (Teubner, 1904) for article 35, would add to the logical completeness of the proofs, and a few slight changes might be made in the introductory chapter.

The book contains about 200 well selected problems, and is an admirable text-book. It supplies the need, felt by some, of a text-book for those who wish to become familiar, in a short time, with the fundamental conceptions of the calculus.

EDWARD L. DODD.

Leçons sur les Fonctions Discontinues. Par RENÉ BAIRE. Rédigées par A. DENJOY. Paris, Gauthier-Villars, 1905. 8vo. viii + 127 pp.

IN these Leçons the Borel series of monographs has given us a work of fundamental importance in a too long neglected field. The interest in discontinuous functions is happily increasing, and finds in this little book a basis for attack and for

classification, so essential if we are not to grope blindly. The following theorem gives the main results:*

A necessary and sufficient condition that a function be the limit of continuous functions is that it be pointwise discontinuous on every perfect assemblage.

Attention should also be called to the concept of semi-continuity, introduced in the proof of this theorem. The limit of the maximum values taken on by a limited function f in intervals (C, D) containing the point A, as (C, D) approaches zero, is called the *maximum* of f at A. If this equals the functional value at the point A, f is said to possess *upper semi-continuity* at A.

The book closes with a general classification of functions. Continuous functions constitute class 0; limits of continuous functions, not themselves continuous, form class 1; in general, limits of functions of class n, not belonging to class n, form class $n + 1$. No attempt is made to show that there exist functions in each class.†

In accordance with the general plan of the series, no advanced mathematical knowledge on the part of the reader is presupposed; this has fortunately compelled the author to give us a very clear and elegant treatment of certain portions of the theories of assemblages and transfinite numbers.

W. D. A. WESTFALL.

Differential Equations. By D. F. CAMPBELL, Armour Institute of Technology. New York, The Macmillan Company, 1906. 96 pp.

THIS is a text-book intended to give a student of engineering a short, practical course in solving differential equations. The author gives only the most common, straightforward methods and omits integrating factors, singular solutions, geometric interpretations, etc.

The first chapter recalls some theorems of algebra and calculus and gives the derivation of a differential equation from its primitive. The use of j instead of i to represent $\sqrt{-1}$ can hardly be considered an improvement. The second chapter deals with changes of the variable (which are practically never used in the rest of the book). It might well have given

*The proof has been simplified by H. Lebesgue, Leçons sur les fonctions de variables réelles par É. Borel, note 2.

†Lebesgue has proved that functions exist in each class, *Journal de Mathématiques*, 1905.

place to a short explanation of the geometric interpretation of differential equations. In chapter 3, equations of the first order and first degree are discussed. Only the more important types are treated — those which are linear, or can be made so; those which are homogeneous in x and y, or can be made so; those in which the variables are separable; and those which are exact.

In the chapter on linear equations with constant coefficients, the usual practice is followed of taking a value of y and proving that it satisfies the equation. In the case of a repeated root of the auxiliary equation, why not follow the same plan? Put $y = xe^{m_1 x}$ and show that the equation is satisfied if m_1 is a multiple root. When the $D - \alpha$ operator is introduced, the author goes more slowly than most writers and spends more time than usual in proving the fundamental properties. This should be an improvement, from a pedagogic standpoint. A number of practical examples are added at the end of this chapter. A solitary use for chapter 2 is found in the type of equation so unfortunately called "homogeneous linear." These, together with equations lacking x or y explicitly, are all the equations of order higher than the first which are dealt with. A short chapter is inserted dealing with integrable equations containing more than two variables, and the book closes with a chapter on partial differential equations. No attention is given to systems of equations save that two sample problems are worked; only six examples are given to be solved, so that the last chapter hardly justifies its presence.

Throughout the book the exercises are collected at the end of the chapters, but are arranged in classes, thus giving the student little practice in detecting the type to which any particular equation belongs. More might have been added in a number of places.

A number of misprints have been allowed to creep in. In § 12, c^a should be ce^a; in § 57, $z = 0$ should be $u = c$; at the end of § 65, "linear" should be omitted; the "familiar theorem of algebra" in example 2, § 69, needs amending. On page 33, $\sin x\,y$, $\cos x\,y$, etc., are used when it would be better to put $y \sin x$, $y \cos x$, etc.

C. R. MacInnes.

Differential Equations. By A. COHEN, Johns Hopkins University. New York, D. C. Heath and Company, 1906. 270 pp.

WHAT from the outside looks like a small text book on differential equations turns out to be a much fuller discussion of the subject. By using good thin paper and eliminating unnecessary margin space, the author has given, in convenient size, quite as full a treatment as either Johnson or Murray. The mechanical work has been well done; the book is well printed, well bound and the proofreading has been carefully done. Even the answers to the problems are accurate as far as they have been tested. A very useful feature of the book is a summary at the end of each chapter of the ground covered in the chapter.

The book opens in the usual way with a chapter on the formation of differential equations. Then equations of the first order and their geometric interpretation are taken up. Before going into equations of order higher than the first, the author introduces (with a question mark) a chapter on total differential equations with three or more variables. Many will feel like emphasizing the question mark. The author's arrangement has the advantage of giving the student miscellaneous exercises of the first order and first degree. It should tend to systematize his knowledge of this fundamental class.

In the chapter on linear equations with constant coefficients, there is a pretty general discussion of the methods of finding the particular integral. In addition to the two ways of breaking up $1/f(D)$, there are also the methods of variation of the parameters and of undetermined coefficients. No mention is made of the short methods of evaluating $e^{ax}/f(D)$, $\sin ax\,/f(D^2)$, etc., these being replaced by the method of undetermined coefficients. In this Dr. Cohen has taken a step forward. As treated by him, any linear equation with constant coefficients can be solved without the use of integration, provided the right hand member is made up of terms having a finite number of distinct derivatives. It is hardly as immediate as one or two of the short methods, but the advantage of having only a single method to carry in mind counterbalances this. It also puts into one general class all the equations to which the method is applicable.

Very little is given on second order equations, most of the special methods for these being in the chapter on equations of

any order. A good chapter on integration in series is brought in, though it might have been moved forward to let the chapter on systems of equations come closer to partial differential equations.

The book would have been improved, I think, by the addition of some easier examples. Those given work out very nicely but require some skill and accuracy in algebra. No misprints of any consequence were found save on page 15, Example 6, *ay* is put instead of *dy*. Many interesting little historical notes have been inserted and references have also been added freely. C. R. MacInnes.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. James. John Wiley and Sons, New York, 1905. xii + 171 pp.

On page 102 of this excellent little book the author states its object in the following words: "It is my purpose here to develop in a rigorous manner the elementary theory of the motion of a *material point* or *particle*, and to thereby furnish a point of departure for the consideration of the motion of bodies as they actually occur in the material universe. This elementary exposition will then serve as an introduction to that portion of the subject which is known as *rational mechanics*, and in which the mathematical theory of the motion of portions of matter of *ideal* forms is investigated under *ideal* conditions, leaving the special applications to the particular sciences." The first eight chapters (pages 1–100) are devoted to the kinematics of a point and the last chapters IX to XVII to the mechanics of a free particle. It is not necessary to give in detail the contents of each chapter, some of which are extremely short; it will suffice to state the general structural lines along which the author has laid out the contents of the book. These lines are kept adroitly before the eyes of the reader, who is led straight to the goal and does not lose himself in the detail of side issues. The geometric derivative of a vector and its projection on an arbitrary axis is the fundamental concept upon which velocity and acceleration and their components depend. They are studied in detail both in rectilinear and curvilinear motion, where in each case proper distinction is made between absolute and relative motion; this is next extended to angular and axial motion, where again the same notions come into use which have been developed under linear motion. Here too the peri-

odic form of motion is gone into in some detail. In the last two chapters of this preliminary kinematic study the motion is referred to coordinate axes in space, which are either fixed or movable. Before the second part of the book is discussed a few remarks concerning the first are offered: In the definition of the geometric derivative of a vector the figure 10 is somewhat misleading to a beginner inasmuch as the tangent PP' is here perpendicular to the vector AP which is not necessary. Since the parallelogram of vectors is the foundation of the entire structure it would have helped the understanding to proceed with the derivative of the vector with moving origin, as Budde and others have done. This secures to the beginner a geometric insight and guide which in the study of mechanics is all important. For this reason it would have been desirable to go into the velocities and accelerations for polar coordinates from the standpoint of movable axes and to make plain to the reader the significance of the expression $d^2r/dt^2 - r(d\theta/dt)^2$. The corollary of article 12 which plays such an important part in the book would have gained in lucidity and importance by such a geometric study. In this way the fact that the acceleration vector lies in the osculating plane would have come out with full force and not as a supplementary algebraic result (see §64). For the same geometric reason the notion of areal velocity and areal acceleration might have been replaced by their usual equivalents, especially since the dimensions of these terms differ from those of linear velocity and linear acceleration; in the same way angular velocity and acceleration are concepts which in the mechanics of a particle should have but transitory importance.

The characteristic feature of the second part of the book is found in the fact that the potential (or as the author puts it the field of force) is considered as the primary mechanical concept. Force as such is merely used as an abbreviation in the text. Newton's three fundamental principles are expressed in a fashion consistent with the four fundamental notions of space, time, mass, and field of force. It is to be supposed that the author will publish additional volumes in which the wisdom of employing the field of force as a primary concept in a rather elementary text is made more cogent than would appear from the limited material offered in this book. In conclusion it must be stated again that although the reader might differ from the author in many particulars as to choice of subject

matter and form of presentation, he cannot help enjoying the clearness, consistency, and cogency with which the author presents his case from cover to cover.

KURT LAVES.

Cours d' Astronomie. Première Partie: Astronomie théorique. By H. ANDOYER. Paris, Librairie Scientifique A. Hermann, 1906. 222 pp.

AFTER a rather prolonged lull in the publication of textbooks on spherical and practical astronomy we are now about to to receive from the press two treatises on the same subject, the one by Professor W. Foerster, the former eminent director of the observatory of Berlin, the other by Professor Andoyer, the well-known scholar of the Paris observatory. The first parts of both treatises have just left the press. A review of Professor Foerster's book has been published in the *Astrophysical Journal.* In giving a short outline of Professer Andoyer's book in a journal devoted to mathematical science, an effort is made to bring out those points primarly which are of interest to mathematicians. Chauvenet's two large volumes on spherical astronomy are too voluminous to lend themselves easily to the needs of a mathematician who tries to inform himself about the application made in astronomy of a certain mathematical theorem he is interested in. Andoyer's book is much more adapted for such purposes. After an introductory chapter concerning spherical trigonometry and a short deviation into spheroidal trigonometry to the extent to which this is needed for elementary geodesic questions, the author gives in the next seven chapters a rather condensed account of refraction, parallax and aberration. Before the theory of precession and nutation is taken up, the reader is initiated in Chapter IX into the more elementary notions of celestial mechanics. This is necessary since the apparent position of a planet, after it has been corrected for refraction, parallax, and aberration, will yet have to be freed from the disturbing influences of the neighboring bodies. To quote but one example: in the definition of apparent solar time, the center of the earth is pulled out of its elliptic path by the various members of the solar system. These perturbations of the individual members must be brought into tables so that for a given value of the time the amount of pull due to each individual member can be properly added to the position of the earth in the elliptic path. Now since the

apparent solar time is measured in the ecliptic, from the point of vernal equinox, *i. e.*, the point where the equator and the ecliptic of a time intersect each other in springtime, it is obvious that the perturbing influence of surrounding matter on the oblate earth is to affect the positions of the planes of reference: equator and ecliptic. These changes are comprised under the name of precession and nutation. In the ninth chapter a short discussion is found concerning the convergence of series used in astronomy. It is the only place in the book where a knowledge beyond the differential and integral calculus is needed. Chapter X is devoted to precession and nutation and formulas are derived for the equatorial and elliptic coordinates to correct the position of an object for the secular and periodic changes in the position of the planes of reference. It would seem as if by the study of the polar triangles the author might have accomplished his result even more successfully, thus keeping the distinction between planetary and luni-solar precession clearly before the eyes of the reader.

The last chapters of the book deal with the geocentric motions of the sun, moon, planets and their satellites. Here the eccentricities and inclinations are assumed to be zero to simplify the deductions. An elementary exposition of the theory of eclipses of moon and sun and of the occultation of stars by the moon finally ends this first part of Professor Andoyer's treatise.

It is perhaps not out of place to point to the great wealth of problems that the subject matter of spherical astronomy affords. In doing so it is but necessary to state in its most general aspect the problem that this science aims to solve: An observer O is located on the surface of a rotating spheroid (earth) the center of which revolves in a well defined manner about the center of gravity of the solar system. A ray of light emanating at the time t_0 from a movable point P (a planet) passes through the atmosphere of the earth and reaches the retina of the observer at the time t_1. We know the heliocentric positions of P and the center of the earth at any time t and we ask to find the *apparent* geocentric position of P at an arbitrarily chosen epoch T. The solution of this problem can be obtained only by putting a number of restrictions on it; to mention but one, we have to keep the interval of time $T - t$ sufficiently small in order to hold the difference between the observed and computed position below an assigned quantity. The contents of the various chapters of the book in question furnish in

turn the steps by which the solution of this problem is accomplished. KURT LAVES.

Vorlesungen über technische Mechanik. Von AUGUST FÖPPL. Dritte Auflage. Bd. 1: *Einführung in die Mechanik*, xvi + 428 pp.; Bd. 3: *Festigkeitslehre*, xvi + 434 pp. Leipzig, B. G. Teubner, 1905.

THE valuable and highly popular work of Föppl on technical mechanics, which began its publication about ten years ago and rapidly ran into a second edition, is now appearing in a third edition of which the first and third volumes are already printed. The present plan calls for no serious changes in the text * other than an expansion into five volumes, the last of which is to contain a considerable amount of matter important for students of technical mechanics but somewhat more advanced than properly finds a place in the general fundamental lectures which fill the first four volumes.

The first or introductory volume still adheres closely to the original maxims of the author, namely, that mechanics is in reality a branch of physics and should be thus presented to beginners, and that no material should be inserted merely because some persons of special and restricted point of view might call for it. One of the most admirable features of the volume is its presentation of the elements of elasticity and hydromechanics in addition to the discussion of the mechanics of a particle and rigid body. It is probably true, and as such it is certainly regrettable, that the great majority of students leave the subject of mechanics after a course by no means meagre with the conviction that mechanics means merely the equilibrium and motion of a particle or rigid body. There is no great difficulty in giving even beginners a realizing sense of the fact that the subject is broader, that the general laws are equally applicable to the study of continuous distributions of matter. This, however, can only be accomplished by thorough adherence to the principle that mechanics is physics rather than mathematics and by a sacrifice of problems which involve complicated mathematical treatment. This the author does with great discretion. His students need not be experts at calculus and analytic geometry to be able to follow with

* An extended review of the four volumes as they appeared in the second edition was given in the BULLETIN, volume 9, pp. 25-35, 1902.

appreciation the treatment of questions taken up in the introductory volume.

Even in the third volume, which especially deals with elasticity, the analysis is kept in the background wherever possible, although toward the close where the subject of wave motion in elastic media is treated and considerable mathematical machinery becomes necessary there is no shrinking from mathematical complications. Such choice in analysis, setting it aside where it can well be done away and unhesitatingly introducing it where it becomes advisable, is one of the most valuable characteristics of the book — and the more valuable as it is so rare in the majority of books with which we are acquainted. Notwithstanding the introduction of some new material the size of the volume has been diminished by eighty pages. The reduction has been possible partly through the elimination of material that seems less vital, partly through the reservation of some subjects for the contemplated fifth volume. The appearance of this supplementary part of the whole work will be awaited with interest by all who are concerned with the question of the proper presentation of mechanics as a whole.

E. B. Wilson.

Carl Friedrich Gauss Werke. Band VII. Theoria Motus Corporum Cœlestium in Sectionibus Conicis Solem Ambientium. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Leipzig, B. G. Teubner, 1906. Pp. 650.

It was within two or three years of a century ago that Gauss's famous Theoria motus first appeared. Now it comes out as the first 290 pages of the seventh volume of his complete works. The remaining 360 pages of the volume are made up of various notes and letters, in small type, which have been culled from the huge Nachlass. To rank all this matter as notes would, however, be extreme minimization; for there are two extensive investigations on the perturbations of Ceres and Pallas filling respectively 35 and 200 pages. Of these the latter for a long time seemed destined to receive the large prize offered by the French academy for a treatment of the perturbations of the asteroid Pallas; but like so much of the work of this Princeps mathematicorum, it never came to publication during his life, which lasted some twenty years after the investi-

gation was practically completed. Nothing shows more clearly the great assiduity and rapidity of Gauss's researches than the way he carried through his computations on Pallas to the extent of over 340,000 figures in the time of about three months.

With this volume VII the edition of Gauss's works, which has occupied nearly forty years in publication, is complete except for one miscellaneous volume, the tenth, which will probably soon be given to the public. The scientific world owes a deep debt of gratitude to those who have so cheerfully spent a large amount of their time and energy on this great and highly valuable undertaking.

E. B. WILSON.

NOTES.

THE July number (volume 29, number 3) of the *American Journal of Mathematics* contains the following papers: "Concerning a certain type of continued fractions depending on a variable parameter," by T. E. MCKINNEY; "Twisted curves whose tangents belong to a linear complex," by V. SNYDER; "Groups in which every subgroup is either abelian or dihedral," by G. A. MILLER; "Lines of curvature of a surface," and "The ovals of the plane sextic curve," by J. E. WRIGHT.

AT the meeting of the London mathematical society held on May 9, the following papers were read: By H. F. BAKER, "Rational expression of the invariants of a quintic by means of three"; by H. LAMB, "Secular stability"; by F. J. W. WHIPPLE, "A lemma connected with Fourier's series."

AT the meeting held on June 13 the following papers were read: By A. R. FORSYTH, "Note on a special set of classes of partial differential equations of the second order"; by T. J. I'A. BROMWICH, "Various extensions of Abel's lemma"; by J. W. L. GLAISHER, "On the number of representations of a number as a sum of $2r$ squares, when $2r$ does not exceed 18"; by A. E. WESTERN, "An extension of Eisenstein's law of reciprocity"; by A. B. BASSET," "On certain singular points of surfaces"; by E. B. ELLIOTT, "The minimum necessary postulates as to a function to be defined as analytic over a region."

THE last list of members of the Société mathématique de France, published in January, contains 280 names, of which 20 are Americans. The meetings of the society are held fort-

nightly except during the summer vacation. The society was founded in 1872, and at once began the publication of its *Bulletin*, the first volume appearing in 1873. A volume has been issued every year since, the parts appearing quarterly. There are 75 journals on its exchange list. Instead of having a special editor, the *Bulletin* is published by the secretaries, who are elected annually. The officers for the present year are: president, Dr. E. BLUTEL; secretaries, Dr. S. SERVANT and Professor L. RAFFY.

THE following mathematical works are announced to be in the press of the Carnegie Institution of Washington and will appear in a few weeks: "Synopsis of linear associative algebra. A report on its natural development and the results reached up to the present time," by J. B. SHAW; "Researches on the performance of the screw propeller," by W. F. DURAND; "Dynamical meteorology and hydrography," by V. BJERKNES and J. W. SANDSTRÖM.

THE following parts of the Encyklopädie der mathematischen Wissenschaften are announced by Teubner of Leipzig to be in the press, and will probably appear during the summer: II, 2, Elliptische Funktionen, by J. HARKNESS and W. WIRTINGER; Automorphe Funktionen, by R. FRICKE; III, 1, Beziehung und Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im neunzehnten Jahrhundert, by G. FANO; Die Gruppentheorie als geometrisches Einteilungsprinzip, by G. FANO; IV, 1, Die elementare Dynamik, by P. STÄCKEL, with additions by J. PETERSEN; IV, 2, Theorie des Schiffes, by A. KRILOFF; Grundlegung der mathematischen Elastizitätslehre, by C. H. MÜLLER and A. TIMPE; Specielle Ausführungen zur Statik elastischer Körper, by O. TEDONE and A. TIMPE; Schwingungen elastischer Körper, insbesondere Akustik, by H. LAMB; VI, 1, Höhere Geodäsie, by P. PIZZETTI.

Of the French edition of the encyclopædia, $I_{1,\,(2)}$ containing Nombres irrationnels et notion de limite, and algorithmes illimités by A. PRINGSHEIM and J. MOLK have just appeared. The details of the editorship of IV, mechanics, have been made public; it is to be published in five volumes under the general direction of Professors Appell and Molk.

THE mathematical and physical section of the Royal society of Naples announces the following prize problem: "A syste-

matic exposition of the results thus far obtained on geometric configurations of the plane and of higher spaces, putting them in relation with the theory of substitutions, and containing if possible some new contribution." Competing memoirs must be written in Italian, Latin or French, and submitted to the secretary not later than June 30, 1908. The prize of 500 lire will be awarded on the first Sunday of 1909.

THE Royal academy of Belgium announces the following prize problems for 1908:

1. Make a systematic exposition of researches already made of critical phenomena in physics. Complete our knowledge of this question by new researches. Prize of 800 francs.

2. An important contribution to the study of the differential equation $Xdx + Ydy = 0$ is desired, X, Y being known quadratic functions of x, y. Prize of 800 francs.

3. New researches on the calorific conductibility of liquids and solutions is desired. Prize of 800 francs.

4. Make a historical and critical study of the experiments on unipolar induction of Weber, and elucidate by means of new experiments, the laws and the interpretation of the physical fact. Prize of 800 francs.

5. Systematize and complete the investigations made in the calculus of variations since 1850. Prize of 600 francs.

Memoirs should be written in French or Flemish and sent to the secretary before August 1, 1908.

THE prize problem of the Prince Jablonowski society of Leipzig for 1910 is as follows: "Most problems in electrostatics are reducible to the determination of the Green distributions of masses, and consequently these distributions are of primary importance for the theory of electrostatics and for the entire potential theory. From recent memoirs (*Leipziger Berichte* for 1906, pages 483–558) it can doubtless be concluded that in the theory of the logarithmic potential for any closed curve the two Green distributions corresponding to the inner and outer spaces can be reduced to a single distribution, the so-called 'fundamental distribution,' and that an analogous property exists in the theory of the newtonian potential for any closed surface. However, in the memoirs mentioned, much is still left to be desired; accordingly the society proposes the following problem: A memoir is desired, in which the theory of the 'fundamental distribution,' either in clearness

and rigor or in extent and completeness, shall be considerably advanced." The prize is 1,500 marks.

THE following advanced courses in mathematics are announced for the academic year 1907–1908:

UNIVERSITY OF CHICAGO (October 1 to June 15). The total number of hours is indicated. — By Professor E. H. MOORE: Selected chapters in analysis, 48 hours; Theory of functions of a complex variable, 24 hours; Seminar, theory of functions of a real variable, 24 hours. — By Professor O. BOLZA: Advanced integral calculus, 96 hours; Calculus of variations, 96 hours. — By Professor H. MASCHKE: Solid analytics and determinants, 48 hours; Algebraic analysis, 48 hours; Projective synthetic geometry, 48 hours; Differential geometry, 96 hours; Partial differential equations, 48 hours. — By Professor L. E. DICKSON: Finite groups with applications to algebra and linear substitution groups, 96 hours. — By Professor H. E. SLAUGHT: Differential equations with applications, 48 hours. — By Professor J. W. A. YOUNG: Introduction to the theory of numbers, 48 hours. — By Dr. A. C. LUNN: Analytic mechanics, 48 hours. — By Professor K. LAVES: Analytic mechanics, 96 hours. — By Professor F. R. MOULTON: Introduction to celestial mechanics, 96 hours; Planetary perturbations, 96 hours.

HARVARD UNIVERSITY. — By Professor W. E. BYERLY: Differential and integral calculus (second course), three hours; Trigonometric series, three hours (with Professor Peirce). — By Professor B. O. PEIRCE: Hydromechanics, three hours. — By Professor W. F. OSGOOD: Elements of mechanics, three hours; Infinite series and products (first half year), three hours; Theory of functions of a complex variable (second half year), three hours. — By Professor M. BÔCHER: Introduction to modern geometry and modern algebra, three hours; Vector analysis and quaternions, three hours; The properties of polynomials (first half year), three hours; Definite integrals and integral equations (second half year), three hours. — by Professor C. L. BOUTON: Elementary theory of differential equations (second half year), three hours: Geometric transformations, three hours. — By Professor J. K. WHITTEMORE: Theory of functions I, three hours; Theory of the figure of the earth (second half year), three hours. — By Dr. J. L. COOLIDGE: Algebraic plane curves, three hours.

UNIVERSITY OF ILLINOIS.— By Professor S. W. SHATTUCK: Differential equations and calculus of variations, three hours. — By Professor E. J. TOWNSEND: Theory of functions, three hours; Seminar, two hours. — By Professor G. A. MILLER: Theory of numbers, three hours; Theory of determinants, two hours. — By Professor H. L. RIETZ: Theory of averages and actuarial theory, three hours. — By Professor J. STEBBINS: Method of least squares, two hours. — By Professor C. N. HASKINS: Solid analytic geometry, three hours; Spherical harmonics and the potential function, three hours. — By Miss M. B. WHITE: Teacher's course, two hours. — By Dr. L. I. NEIKIRK: Theory of equations, three hours. — By Dr. C. H. SISAM: Modern geometry and algebraic surfaces, three hours. — By Dr. A. R. CRATHORNE: Partial differential equations, two hours.

INDIANA UNIVERSITY. — By Professor R. J. ALEY: Theory of numbers, two hours; Differential equations, three hours (autumn, winter); Mathematical pedagogy, two hours (s) — By Professor S. C. DAVISSON: Modern analytic geometry, two hours (a, w); Theory of surfaces, two hours; Non-euclidean geometry, two hours (w, s). — By Professor D. A. ROTHROCK: Advanced calculus, three hours; Quaternions with applications, three hours (a, w); Potential functions, two hours (w, s). — By Professor U. S. HANNA: Substitution groups, three hours (a); Galois theory of equations, three hours (w). — By Dr. C. HASEMAN: Partial differential equations, three hours.

OXFORD UNIVERSITY (*Michaelmas Term*). — By Professor W. ESSON: Analytic geometry of plane curves, two hours; Synthetic geometry of plane curves, one hour. — By Professor E. B. ELLIOTT: Sequences and series, two hours; Elementary theory of numbers, one hour. — By Professor A. E. H. LOVE: Magnetism and electricity, three hours. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours. — By Mr. E. H. HAYES: Analytic statics, two hours. By Mr. A. L. DIXON: Hydrostatics, one hour. — By Mr. H. T. GERRANS: Tridimensional rigid dynamics, two hours. — By Mr. C. E. HASELFOOT: Theory of equations, one hour. — By Mr. P. J. KIRKBY: Projective geometry, two hours. — By

Mr. A. E. JOLLIFFE: Analytic geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. R. F. MCNEILE: Curve tracing, one hour. — By Mr. C. H. SAMPSON: Solid geometry, one hour.

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Lebesgue's integrals, Dirichlet's principle, Fourier's series, three hours. — By Professor L. DONATI: Electromagnetic fields and dynamics of electrons, three hours. — By Professor L. PINCHERLE: Analytic functions; algebraic functions and their integrals, three hours.

UNIVERSITY OF CATANIA. — By Professor G. LAURICELLA: Theory of heat and propagation of waves, four and a half hours. — By Professor G. PENNACCHIETTI: Advanced kinematics and rigid dynamics, four and a half hours. — By Professor M. PIERI: Projective geometry of hyperspace, three hours. — By Professor C. SEVERINI: Finite continuous groups of transformations, contact transformations, four and a half hours.

UNIVERSITY OF GENOA. — By Professor G. FUBINI: Existence theorems of the calculus of variations, three hours. — By Professor G. LORIA: Plane representation of algebraic surfaces, rational transformations of the plane and space, three hours. — By Professor O. TEDONE: Spherical functions, Lamé's functions with applications to electro- and magnetic statics, three hours.

UNIVERSITY OF MESSINA. — By Professor G. BAGNERA: Theta functions of several arguments and relative groups of characteristics, three hours. — By Professor R. MARCOLONGO: Electric and optical phenomena in moving media, three hours. — By Professor V. MARTINETTI: Projective geometry of hyperspace, three hours — By Professor L. ORLANDO: Definite integrals with applications to mathematical physics, three hours; Principles of the theory of numbers, two hours. — By Professor G. VIVANTI: Calculus of variations, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics during the last three centuries, three hours. — By Professor A. CAPELLI: Theory of groups with analytic applications, four and a half hours. — By Professor P. DELL PEZZO: Analytic functions, Riemann surfaces and automorphic functions, four and a half hours. — By Professor D. MONTESANO: Theory of hyperspaces and birational

transformation, four and a half hours. — By Professor L. PINTO: Physical optics, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Differential equations and theory of functions of a complex variable, four and a half hours. — By Professor A. FAVARO: History of Italian mathematics in the sixteenth and seventeenth centuries, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Differential equations of dynamics; contact transformations with applications to dynamics and optics, four and a half hours. — By Professor G. RICCI: Theory of potential, electro- and magneto-statics, four hours. — By Professer F. SEVERI: Advanced theory of algebraic functions of one and two variables, three hours. — By Professor G. VERONESE: Geometry of hyperspace, four hours.

UNIVERSITY OF PALERMO. — By Professor F. GERBALDI: Differential geometry, four and a half hours. — By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and a half hours. — By Professor G. TORELLI: Mathematical theory of elasticity, four and a half hours. — By Professor A. VENTURI: Rotation of solid bodies, application to the earth, precession, nutation and motion of the pole, four and a half hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI: Laplace's equation with applications to theoretical physics, three hours. — By Professor L. BERZOLARI: Algebraic forms with geometric applications, three hours. — By Professor E. PASCAL: Contact transformations with applications, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Hyperspaces, geometry on an algebraic curve with applications, four and a half hours. — By Professor L. BIANCHI: Differential geometry of curves and surfaces, surfaces applicable to quadrics, four and a half hours. — By Professor U. DINI: Harmonic functions and Fourier's series, four hours. — By Professor G. A. MAGGI: Equilibrium and motion of elastic solid bodies, with applications to optics, four and a half hours. — By Professor P. PIZZETTI: General theory of planetary perturbations, three hours.

UNIVERSITY OF ROME. — By Professor G. BISCONCINI: Mathematical theory of elasticity, with technical applications,

three hours. — By Professor G. CASTELNUOVO: Geometry on an algebraic surface, three hours. — By Professor V. CERRUTI: Calculus of variations, with applications to geometry and mechanics, three hours. — By Professor V. VOLTERRA: Electromagnetic fields, four and a half hours: The problem of three bodies, three hours.

UNIVERSITY OF TURIN. — By Professor T. BOGGIO: Applications of Fredholm's integral equations to mathematical physics, three hours. — By Professor E. D'OVIDIO: Theory of algebraic forms, three hours. — By Professor G. MORERA: Newtonian forces, equilibrium of a rotating liquid, three hours. — By Professor C. SEGRE: Selected chapters of line geometry, three hours. — By Professor C. SOMIGLIANA: Propagation of heat, kinetic theory of gases, three hours.

WITH the aid of a gift from Mr. C. S. BARTON, Columbia University has secured a very extensive collection of mathematical models and instruments for a mathematical laboratory and museum.

IN Erlangen, on the twenty-seventh of April, the seventieth birthday of Professor PAUL GORDAN was appropriately celebrated by an elaborate programme, including numerous addresses from associates and collaborators. Professor Gordan will retire from active service at the close of the present semester.

PROFESSOR E. SELLING, of the University of Würzburg, has retired from active teaching.

MR. ARTHUR HOLDEN has been appointed assistant lecturer and tutor in mathematics at the University of Sheffield.

THE Accademia dei Lincei of Rome, has divided the royal prize of 10,000 lire for mathematics equally between Professor C. ARZELÀ, of the University of Bologna, and Professor G. CASTELNUOVO, of the University of Rome.

THE Italian scientific society has awarded its gold medal for researches in mathematics to Professor G. LAURICELLA, of the University of Catania.

MR. H. BATEMAN, lecturer in the University of Liverpool, has been appointed reader in mathematical physics at the University of Manchester. MR. J. E. LITTLEWOOD has been appointed lecturer in mathematics and MR. H. M. PRIESTLEY assistant lecturer in mathematics in the latter university.

At the University of Michigan, Professor J. L. Markley has been promoted to a full professorship of mathematics. Dr. P. Field, Dr. W. B. Ford and Dr. T. R. Running have been promoted to assistant professorships of mathematics. Dr. J. W. Glover, professor of mathematics and insurance, has spent the past year, on leave of absence, as consulting actuary of the Wisconsin legislative committee on banks and insurance.

Dr. C. C. Grove, of Hamilton College, has been appointed professor of mathematics in Roanoke College.

At the Massachusetts Institute of Technology, Professor F. H. Bailey has been promoted to a full professorship of mathematics, and Mr. L. M. Passano to an assistant professorship.

Professor H. F. Stecker, of Pennsylvania State College, has been promoted to an associate professorship of mathematics.

At Cornell University Mr. C. F. Craig and Dr. F. W. Owens have been appointed instructors in mathematics.

At the University of Illinois, Dr. A. R. Crathorne has been appointed instructor in mathematics, and Mr. H. P. Kean, and Mr. W. W. Denton assistants in mathematics. Dr. E. L. Dodd has been appointed instructor in mathematics at the University of Texas.

Dr. A. L. Underhill, of Princeton University, has been appointed instructor in mathematics at the University of Wisconsin.

Mr. Joseph Lipke, of Columbia University, has been appointed instructor in mathematics at the University of California.

At Vassar College, Miss E. Morenus has been appointed instructor in mathematics; Miss G. Smith has been granted a leave of absence and will spend next year in study at Paris.

Dr. Edward John Routh, the well-known mathematician of Cambridge, died June 7, 1907, at the age of 76 years. He was author of several works on dynamics and formerly lecturer of mathematics at Peterhouse aud Pembroke Colleges.

Professor Francesco Siacci, senator of the kingdom of Italy and professor of theoretical mechanics at the University of Naples, died May 31, at the age of 68 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALBRICH (K.). Behandlung der Funktionen im Mittelschulunterricht. (Progr.) Hermannstadt, 1906. 8vo. 47 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Herausgegeben im Auftrage der Akademien der Wissenschaften zu Göttingen, Leipzig, München und Wien, sowie unter Mitwirkung zahlreicher Fachgenossen. Band V: Physik. Redigiert von A. Sommerfeld. Teil 2, Heft 2. Leipzig, Teubner, 1907. 8vo. Pp. 289–392. M. 3.00

GALDEANO (Z. G. de). Tratado de analisis matematica. Tomo V: Applicaciones del calculo infinitesimal al estudio de las figuras en el espacio. Zaragoza, 1906. 4to. 576 pp.

GARBIERI (G.). Geometria analitica. Riassunto di lezioni date nell' Università di Genova. Parte I: Luoghi di primo grado, con cenno su altri luoghi in coordinate Cartesiane. Torino, 1907. 8vo. 60 pp.

KLEIN (H.). Die Dreiersysteme von 13 Elementen. (Progr.) Hermannstadt, 1906. 8vo. 23 pp.

KOSMIK (K.). Pythagoreische Dreiecke und diesbezügliche Vierecke. (Progr.) Mähr.-Weisskirchen, 1906. 8vo. 41 pp.

KRUG. Die niedere Analysis auf der Unterrichtsstufe des Realgymnasiums. Teil 2. (Progr.) Stuttgart, 1906. 4to. 69 pp.

LORENTZ (H. A.). Lehrbuch der Differential- und Integralrechnung, nebst Einführung in andere Teile der Mathematik mit besonderer Berücksichtigung der Bedürfnisse der Studirenden der Naturwissenschaften. Uebersetzt von C. G. Schmidt. 2te Auflage. Leipzig, 1907. 8vo. 6 + 562 pp. M. 12.00

LUBELZA (J.). Lecciones de cálculo infinitesimal. Libros II y III: Cálculo integral y aplicaciones. Madrid, 1906. 4to. Pp. 1–22 and 131–382.

MELLOR (J. W.). Higher mathematics for students of chemistry and physics. New York, Longmans, 1906. 8vo. Cloth. $4.50

MÖBIUS (P. J.). Ueber die Anlage zur Mathematik. 2te vermehrte und veränderte Auflage. Leipzig, 1907. 8vo. 16 + 264 pp. M. 4.50

HILL (G. W.). The collected mathematical works. Vol. 4. Washington, D. C., Carnegie Institution, 1907. 4to. 6 + 460 pp. Cloth. $2.50

HUNTINGTON (E. V.). La kontinuo. Elementa teorio starigita sur la ideo de ordo kun aldono pri transfinitaj nombroj. Tradukita de la anglo lingvo kun la permeso de la aŭtoro de Raoul Bricard. Paris, Gauthier-Villars, 1907. 16mo. 10 + 126 pp. F. 2.75

PAPELIER (G.). Précis de géométrie analytique à l'usage des élèves de mathématiques spéciales. 2e fascicule. Paris, Vuibert, 1907. 8vo. Pp. 449 à 696.

ROTH (C.). Einiges über ältere und neuere Methoden der Rektifikation sowie der Teilung von Kreisbögen. (Progr.) Mühlbach, 1906. 4to. 22 pp.

SALMON (G.). Analytische Geometrie der Kegelschnitte mit besonderer Berücksichtigung der neueren Methoden. Nach Salmon frei bearbeitet von Wilhelm Fiedler. 7te Auflage. Teil I. Leipzig, Teubner, 1907. 8vo. 35 + 444 pp. Cloth. M. 10.00

SIEVERT (H.). Die Parameterdarstellung der Kurven 3ter Ordnung durch Thetafunktionen. (Progr.) Bayreuth, 1906. 8vo. 43 pp.

TRAYNARD (C.). Sur la fonction thèta de deux variables et les surfaces hyperelliptiques. (Thèse.) Paris, Gauthier-Villars, 1907. 4to. 107 pp.

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, vorzugsweise für Realschulen, Progymnasien und Realprogymnasien. Leipzig, Teubner, 1906. 8vo. 7 + 315 pp. M. 2.60

BONET y GARCIA (J.). Apuntes de algebra elemental. Segunda edición corregida y aumentada. 2 vols. Toledo, Menor, 1906. 12mo. 251 pp. P. 7.00

BOS (H.). Geometria elemental. Paris, Hachette, 1906. 16mo. 287 pp.

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales. Table numérique. Tables trigonométriques. I, Division centésimale; II, Division sexagésimale établies conformément à l'arrêté ministériel du 3 août, 1901, à l'usage des candidats au baccalauréat et aux écoles polytechnique et de Saint-Cyr. 6e édition. Paris, Hachette, 1907. 8vo. 176 pp.

BRUNO (G. M.). Geometria. Curso superior con numerosos ejercicios y nociones de agrimensura, levantamiento de planos y nivelación. Paris, 1907. 8vo. 368 pp.

CIAMBERLINI (C.). Elementi di geometria per le scuole tecniche. Torino, Paravia, 1907. 8vo. 132 pp. L. 2.00

—— Algebra pratica per le scuole tecniche. Torino, Paravia, 1907. 8vo. 63 pp. L. 1.00

FERVAL (H.). Eléments de trigonométrie rédigés conformément aux programmes de l'enseignement secondaire et de l'enseignement primaire supérieur, contenant 556 exercices et problèmes. 3e édition, corrigée et conforme aux programmes du 27 juillet 1905. Paris, Hachette, 1907. 16mo. 304 pp. F. 2.50

FRANCO CABALLERO (F.). Principios de geometria, dispuestos para uso de las escuelas primarias de niños y niñas. Barcelona, Carbonell, 1905. 12mo. 179 pp. P. 1.07

GEYFFARTH (W.). Allgemeine Arithmetik und Algebra. Zum Gebrauche an höheren Lehranstalten herausgegeben. 3te Auflage. Dresden-Blasewitz, Bleyl, 1907. 8vo. 8 + 142 pp. M. 1.80

—— Trigonometrie. Zum Gebrauche an höheren Lehranstalten und zum Selbstunterricht. Dresden-Blasewitz, Bleyl, 1907. 8vo. 90 pp. M. 1.60

KNOCHENDOPPEL (C.). Hauptsätze der Arithmetik. Kurze Zusammenfassung der wichtigsten Begriffe und Gesetze. Apolda, Lauth, 1907. 8vo. 24 pp. Boards. M 0.85.

KÜSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. Im Einverständnis mit der Atomgewichtskommission der deutschen chemischen Gesellschaft für den

Gebrauch im Unterrichtslaboratorium und in der Praxis berechnet und mit Erläuterung versehen. 7te verbesserte und vermehrte Auflage. Leipzig, Veit, 1907. 8vo. 106 pp. Cloth. M. 2.40

LEMAIRE (G.). Questions d'algèbre élémentaire. (Homogénéité, symétrie, calcul rapide.) Paris, Vuibert, 1907. 8vo. 189 pp.

MYERS (G. W.). First-year mathematics for secondary schools. Chicago, University Press, 1907. 12mo. 198 pp. Cloth. $1.00

PULLAR (A.). Geometry for kindergarten students. Specially adapted to meet the requirements of the examination of the national Froebel union. London, Sonnenschein, 1907. 12mo. 238 pp. 3s.

RATINET (A.). See BOUVART (C.).

RUBIO (J. M.). Apuntes de Geometria. Madrid, Diaz, 1906. 12mo. 155 pp. P. 3.00

SABRAS y CAUSAPÉ (T.). See SANCHEZ RAMOS (E.).

SANCHEZ RAMOS (E.) y SABRAS y CAUSAPÉ (T.). Nociones e ejercicios de aritmetica y geometria. 4a edicion. Sevilla, 1906. 4to. 206 pp.

SCHULTZ (E.). Trigonometrische Aufgaben aus den Gebieten der Praxis, unter Zugrundelegung der natürlichen Funktionenwerte, für gewerbliche Lehranstalten. Essen, Baedeker, 1907. 8vo. 4 + 52 pp. Boards. M. 1.40

—— Leitfaden der Trigonometrie, unter Zugrundelegung der natürlichen Funktionenwerte, nebst einer Sammlung von Aufgaben aus den Gebieten der Praxis für gewerbliche Lehranstalten. Essen, Baedeker, 1907. 8vo. 8 + 116 pp. Boards. M. 1.40

VERONESE (G.). Nozioni elementari di geometria intuitiva, ad uso dei ginnasi inferiori. 3a edizione. Padova, Drucker, 1906. 8vo. 92 pp. L. 1.00

WALTHER (F.). Lehr- und Uebungsbuch der Geometrie für die Mittelstufe. Mit Anhang. I: Ebene Trigonometrie. II: Abbildung und Berechnung einfacher Körper. Berlin, Salle, 1907. 8vo. 8 + 204 pp. M. 2.20

III. APPLIED MATHEMATICS.

ANTILLI (A.). Manual de dibujo geométrico é industrial. Traducido de la tercera edición italiana y considerablemente aumentado por D. Antonio Llorens y Clariana. Barcelona, Gili, 1907. 12mo. 156 pp. P. 2.50

CARONNET (T.). Formulaire de mécanique (mathématiques spéciales); cinématique, dynamique, statique. Paris, Vuibert. 8vo. 76 pp.

CLARK (D. K.). The mechanical engineer's pocket-book of tables, formulæ, rules and data. 6th edition, revised and enlarged by H. H. P. Powles. London, Lockwood, 1907. 12mo. Leather. 6s.

DALBY (W. E.). Valves and valve-gear mechanisms. New York, Longmans, 1906. 8vo. Cloth. $6.00

FISCHER (C.). Ueber die optische Abbildung. Die Behandlung ihrer geometrischen Theorie in der Schule. (Progr.) Leipzig, Hinrich, 1907. 8vo. 46 pp. M. 1.00

FRENCH (L. G.). Steam turbines, practice and theory. Brattleboro, Technical Press, 1907. 8vo. 6 + 418 pp. Cloth. $3.00

GIORLI (E.). Il meccanico. 5a edizione ampliata. Milano, Hoepli, 1907. 16mo. 15 + 574 pp. L. 2.00

HOVENDEN (F.). Practical mathematics for young engineers. Specially arranged for apprentices who intend to become marine engineers. London, Simpkin, 1907. 12mo. 144 pp. Cloth. 2s. 6d.

HOWE (M. A.). Retaining-walls for earth; including the theory of earth-pressure as developed from the ellipse of stress; with a short treatise on foundations. 4th edition. New York, Wiley, 1907. 12mo. 10 + 167 pp. Cloth. $1.25

HUTCHINSON (R. W.). Long-distance electric power transmission: being a treatment on the hydro-electric generation of energy; its transformation, transmission and distribution. New York, Van Nostrand, 1907. 12mo. 5 + 345 pp. Cloth. $3.00

KRAUSS (F.). Die Thermodynamik der Dampfmaschinen. Berlin, Springer, 1907. 8vo. 8 + 144 pp. M. 3.00

LEBLOND (H.). Mesures électriques. Electricité experimentale et pratique. 3e édition. Vol. 2. Paris, Berger-Levrault, 1907. 8vo. 6 + 616 pp. F. 9.00

MACKROW (C.). The naval architect's and shipbuilder's pocket-book of formulæ, rules and tables, and marine engineer's and surveyor's handy-book of reference. 9th edition. New York, Van Nostrand, 1907. 16mo. 750 pp. Leather. $5.00

NERNST (W.). Experimental and theoretical applications of thermodynamics to chemistry. New York, Scribner, 1907. 8vo. 10 + 123 pp. Cloth. $1.25

POUSSIN (R.). Traité élémentaire des assurances sur la vie. Principes et applications. Paris, Dulac, 1907. 8vo. 560 pp. F. 20.00

RIPOLL. Etude sur les principaux calculs nautiques. Paris, Chapelot, 1907. 8vo. 87 pp.

SCHLESSER (E.). Géométrie descriptive et géométrie cotée. (Classes de première et de mathématiques: Préparation à l'école navale, à l'école spéciale militaire de Saint-Cyr, à l'institut agronomique, etc.) Paris, Delgrave, 1907. 8vo. 243 pp. F. 3.00

SCHLINK (W.). Statik der Raumfachwerke. Leipzig, Teubner, 1907. 8vo. 14 + 390 pp. Cloth. M. 9.00

TAYLOR (H. D.). A system of applied optics, being a complete system of formulæ of the second order, and the foundation of a complete system of the third order, with examples of their practical application. New York, Macmillan, 1906. 4to. Cloth. 30s.

TREVEN (K.). Absolute und praktische Masse und ihr Zusammenhang. (Progr.) Friedek, 1906. 8vo. 42 pp.

SIXTEENTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

AKERS, O. P. On the Congruence of Axes in a Bundle of Linear Line Complexes. Read Dec. 29, 1905. *American Journal of Mathematics*, vol. 29, No. 1, pp. 23–47; Jan., 1907.

ALLARDICE, R. E. On a Limit of the Roots of an Equation that is Independent of all but Two of the Coefficients. Read (San Francisco) Feb. 23, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 9, pp. 443–447; June, 1907.

AMES, L. D. Note on the Orientation of a Secant. Read (Southwestern Section) Dec. 1, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 5, pp. 240–241; Feb., 1907.

BJERKNES, V. F. Experimental Demonstration of Hydrodynamic Action at a Distance. Read Dec. 29, 1905. Included in the author's Fields of Force, New York, 1906.

BLICHFELDT, H. F. On the Order of Linear Homogeneous Groups (Supplementary Paper). Read (Chicago) Apr. 14, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 523–529; Oct., 1906.

—— On Modular Groups Isomorphic with a Given Linear Group. Read (Chicago) Apr. 14, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 30–32; Jan., 1907.

BLISS, G. A. Note on Maxima and Minima of Functions of Several Variables. Read (Chicago) Dec. 28, 1906. *American Mathematical Monthly*, vol. 14, No. 3, pp. 47–49; Mar., 1907.

—— The Construction of a Field of Extremals about a Given Point. Read Feb. 23, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 7, pp. 321–324; Apr., 1907.

BOLZA, O. Weierstrass's Theorem and Kneser's Theorem on Transversals for the Most General Case of an Extremum of a Simple Definite Integral. Read Sept. 4, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 459–488; Oct., 1906.

BRENKE, W. C. On the Differentiation of Trigonometric Series. Read Apr. 28, 1906. *Annals of Mathematics*, ser. 2, vol. 8, No. 2, pp. 87–93; Jan., 1907.

CARMICHAEL, R. D. Multiply Perfect Numbers of Three Different Primes. Read Sept. 3, 1906. *Annals of Mathematics*, ser. 2, vol. 8, No. 1, pp. 49–56; Oct., 1906.

—— On Euler's ϕ-Function. Read Dec. 28, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 5, pp. 241–243; Feb., 1907.

—— A Table of Multiply Perfect Numbers. Read Feb. 23, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 8, pp. 383–386; May, 1907.

—— On a Certain Quartic Curve Which May Degenerate into an Ellipse. Read (Chicago) Mar. 30, 1907. *American Mathematical Monthly*, vol. 14, No. 3, pp. 52–54; Mar., 1907.

CARVER, W. B. Associated Configurations of the Cayley-Veronese Class. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 3, pp. 109–111; Dec., 1906.

COAR, H. L. Functions of Three Real Variables. Read (Chicago) Dec. 29, 1905. *American Journal of Mathematics*, vol. 28, Nos. 3–4, pp. 243–332; July–Oct., 1906.

CURTISS, D. R. Sur certains Théorèmes de la Valeur moyenne. Read Sept. 16, 1904. *Annals of Mathematics*, ser. 2, vol. 8, No. 3, pp. 118–126; Apr., 1907.

—— A Proof of the Theorem Concerning Artificial Singularities. Read (Chicago) Apr. 14, 1906. *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 161–162; July, 1906.

DICKSON, L. E. Linear Algebras in Which Division is Always Uniquely Possible. Read (Chicago) Apr. 14, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 370–390; July, 1906.

—— Criteria for the Irreducibility of Functions in a Finite Field. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 1, pp. 1–8; Oct., 1906.

—— On Commutative Linear Algebras in Which Division is Always Uniquely Possible. Read Sept. 3, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 514–522; Oct., 1906.

—— On the Theory of Equations in a Modular Field. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 1, pp. 8–10; Oct., 1906.

—— Invariants of Binary Forms under Modular Transformations. Read (Southwestern Section) Dec. 1, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 205–232; Apr., 1907.

—— The Symmetric Group on Eight Letters and the Senary First Hypoabelian Group. Read (Chicago) Mar. 30, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 8, pp. 386–389; May, 1907.

EISENHART, L. P. Certain Triply Orthogonal Systems of Surfaces. Read Oct. 28, 1905. *American Journal of Mathematics*, vol. 29, No. 2, pp. 168–212; Apr., 1907.

—— Applicable Surfaces with Asymptotic Lines of One Surface Corresponding to a Conjugate System of Another. Read Sept. 4, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 113–134; Jan., 1907.

FITE, W. B. Irreducible Linear Homogenous Groups Whose Orders are Powers of a Prime. Read Sept. 3, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 107–112; Jan., 1907.

GILLESPIE, D. C. On the Construction of an Integral of Lagrange's Equations in the Calculus of Variations. Read Dec. 29, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 7, pp. 345–348; Apr., 1907.

HASKELL, M. W. The Resolution of Any Collineation into Perspective Reflections. Read Sept. 22, 1904. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 361–369; July, 1906.

HASKINS, C. N. Note on the Differential Invariants of a Plane. Read Apr. 28, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 588–590; Oct., 1906.

HEDRICK, E. R. Note on the Existence of a Continuous First Derivative. Read Aug. 31, 1903. Incorporated in a paper published in the *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 177–192; July, 1906.

—— On the Function $\xi(h)$ in the Law of the Mean. Read (Chicago) Apr. 14, 1906. Incorporated in a paper published in the *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 177–192; July, 1906.

—— On a Final Form of the Theorem of Uniform Continuity. Read Feb. 23, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 8, pp. 378–380; May, 1907.

HUNTINGTON, E. V. Note on the Fundamental Propositions of Algebra. Read Dec. 28, 1905. Incorporated in a paper published in the *Annals of Mathematics*, ser. 2, vol. 8, No. 1, pp. 1–44; Oct., 1906.

HUTCHINSON, J. I. On Certain Automorphic Groups Whose Coefficients are Integers in a Quadratic Field. Read Dec. 28, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 530–536; Oct., 1906.

—— On Loci the Coordinates of whose Points are Abelian Functions of Three Parameters. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 3, pp. 105–109; Dec., 1906.

—— A Method of Constructing the Fundamental Region of a Discontinuous Group of Linear Transformations. Read Dec. 28, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 261–269; Apr., 1907.

KASNER, E. On the Trajectories Produced by an Arbitrary Field of Force. Read April 29, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 401–424; July, 1906.

—— Invariants of Differential Elements for Arbitrary Point Transformation. Read Dec. 28, 1905. *American Journal of Mathematics*, vol. 28, No. 3, pp. 203–213; July, 1906.

—— The Geometry of Dynamical Trajectories. Read September 4, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 135–158; Apr., 1907.

—— Systems of Extremals in the Calculus of Variations. Read Dec. 29, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 6, pp. 289–292; Mar., 1907.

KELLOGG, O. D. The Behavior on the Boundary of Harmonic Functions of a Region. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 4, pp. 168–170; Jan., 1907.

LAVES, K. A Dynamical Interpretation of an Integral of Jacobi's Partial Differential Equation for the Problem of a Solid Body Rotating About a Fixed Point. Read (Chicago) Dec. 29, 1905. *Astronomische Nachrichten*, vol. 171, No. 4095, pp. 225–236; May, 1906.

LEHMER, D. N. On a New Method of Finding Factors of Numbers. Read (San Francisco) Dec. 19, 1903. *Bulletin of the American Mathematical Society*, vol. 13, No. 10, pp, 501–502; July, 1907.

—— On the Orderly Listing of Substitutions. Read (San Francisco) February 24, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 2, pp. 81–84; Nov., 1906.

LENNES, N. J. On the Necessary Conditions for the Convergence of an Improper Definite Integral. Read (Chicago) Dec. 30, 1904. *American Journal of Mathematics*, vol. 29, No. 1, pp. 13–22; Jan., 1907.

—— On Functions of Limited Variation. Read (Chicago) Apr. 14, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 1, pp. 10–20, and No. 2, p. 87; Oct.–Nov., 1906.

LONGLEY, W. R. A Class of Periodic Orbits of an Infinitesimal Body Subject to the Attraction of n Finite Bodies. Read (Chicago) Apr. 14, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 159–188; Apr., 1907.

—— Some Particular Solutions in the Problem of n Bodies. Read Dec. 28, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 7, pp. 324–335; Apr., 1907.

MCDONALD, J. H. The Theory of the Reduction of Hyperelliptic Integrals of the First Kind and of Genus 2 to Elliptic Integrals by a Transformation of the nth Order. Read (San Francisco) Feb. 24, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 578–587; Oct., 1906.

MCKINNEY, T. E. On the Continued Fractions Representing Properly and Improperly Equivalent Real Numbers in a System of Continued Fractions Depending on a Variable Parameter. Read Dec. 29, 1906. *American Journal of Mathematics*, vol. 29, No. 3, pp. 213–278; July, 1907.

MANNING, W. A. On the Primitive Groups of Class Ten. Read (San Francisco) Feb. 25, 1905. *American Journal of Mathematics*, vol. 28, No. 3, pp. 226–236; July, 1906.

—— On Multiply Transitive Groups. Read (San Francisco) Feb. 24, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 499–508: Oct., 1906.

—— A Note on Transitive Groups. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 1, pp. 20–23; Oct., 1906.

MASON, M. On the Boundary Value Problems of Linear Ordinary Differential Equations of the Second Order. Read Sept. 7, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 337–360; July, 1906.

—— Curves of Minimum Moment of Inertia. Read Dec. 29, 1905. *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 165–172; July, 1906.

—— A Necessary Condition for an Extremum of a Double Integral. Read Apr. 28, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 6, pp. 293–298; Mar., 1907.

—— Selected Topics in the Theory of Boundary Value Problems of Differential Equations. Four Lectures. Read Sept. 5–8, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 5, pp. 223–231; Feb., 1907.

MILLER, G. A. Groups in Which Every Subgroup of Composite Order is Invariant. Read (San Francisco) Feb. 24, 1906. *Archiv der Mathematik und Physik*, ser. 3, vol. 11, No. 1, pp. 76–79; Nov., 1906.

—— Generalization of the Groups of Genus Zero. Read Sept. 4, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 1–13; Jan., 1907.

—— Groups of Order p^m Containing Exactly $p+1$ Abelian Subgroups of Order p^{m-1}. Read Oct. 27, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 4, pp. 171–177; Jan., 1907.

—— The Groups in Which Every Subgroup is Either Abelian or Hamiltonian. Read Oct. 27, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 25–29; Jan., 1907.

—— Groups in Which Every Subgroup is Either Abelian or Dihedral. Read (Southwestern Section) Dec. 1, 1906. *American Journal of Mathematics*, vol. 29, No. 3, pp. 289–294; July, 1907.

—— On the Minimum Number of Operators whose Orders Exceed Two in any Finite Group. Read Dec. 29, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 5, pp. 235–239; Feb., 1907.

—— The Groups Generated by Three Operators Each of Which is the Product of the Other Two. Read Feb. 23, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 8, pp. 381–382; May, 1907.

—— Note on the Commutator of Two Operators. Read April 27, 1907. *Bulletin of the American Mathematical Society*, vol., 13, No. 10, pp. 497–501; July, 1907.

MOORE, C. L. E. Geometry of Circles Orthogonal to a Given Sphere. Read Dec. 28, 1905. *Annals of Mathematics*, ser. 2, vol. 8, No. 2, pp. 57–72; Jan., 1907.

MOORE, C. N. On the Introduction of Convergence Factors into Summable Series and Summable Integrals. Read Feb. 23, 1907. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 299–330; Apr., 1907.

MOORE, E. H. The Decomposition of Modular Systems Connected with the Doubly Generalized Fermat Theorem. Read (Chicago) Dec. 29, 1898. *Bulletin of the American Mathematical Society*, vol. 13, No. 6, pp. 280–288; Mar., 1907.

——Note on Fourier's Constants. Read (Southwestern Section) Dec. 1, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 5, pp. 232–234; Feb., 1907.

MORLEY, F. Reflexive Geometry. Read Sept. 4, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 14–24; Jan., 1907.

MORRIS, R. On the Expressibility of the Automorphic Functions of the Group $(0, 3; l_1, l_2, l_3)$ in Terms of Theta Series. Read Sept. 7, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 425–448; July, 1906.

MOULTON, F. R. A Class of Periodic Solutions of the Problem of Three Bodies. Read Sept. 8, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 537–577; Oct., 1906.

—— On the Classes of Periodic Orbits Computed by G. H. Darwin. Read (Chicago) April 14, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 537–577; Oct., 1906.

OSGOOD, W. F. The Calculus in our Colleges and Technical Schools. [Presidential Address.] Read Apr. 27, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 9, pp. 449–467 ; June, 1907.

PEIRCE, G. A New Approximate Construction for π. Read Dec. 29, 1905. *Bulletin of the American Mathematical Society*, vol. 13, No. 4, pp. 166–167 ; Jan., 1907.

PIERPONT, J. On the Area of Curved Surfaces. Read Apr. 28, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 489–498 ; Oct., 1906.

RAGSDALE, VIRGINIA. On the Arrangement of the Real Branches of Plane Algebraic Curves. Read Apr. 29, 1905. *American Journal of Mathematics*, vol. 28, No. 4, pp. 377–404 ; Oct., 1906.

RANUM, A. The Group of Classes of Congruent Matrices and its Application to the Group of Isomorphisms of any Abelian Group. Read Sept. 3, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 71–91 ; Jan., 1907.

—— On Jordan's Linear Congruence Groups. Read Oct. 27, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 7, pp. 336–345 : Apr., 1907.

RICHARDSON, R. G. D. Multiple Improper Integrals. Read Dec. 28, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 449–458 ; July, 1906.

RICHARDSON, SOPHIA F. Note on Poristic Systems of Polygons. Read Oct. 27, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 4, pp. 177–182 ; Jan., 1907.

SAUREL, P. L. On the Distance from a Point to a Surface. Read April 27, 1907. *Bulletin of the American Mathematical Society*, vol. 13, No. 9, pp. 447–448 ; June, 1907.

SCHWEITZER, A. R. Concerning Abstract Geometrical Relations. Read Sept. 3, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 2, pp. 79–81 ; Nov., 1906.

SCOTT, CHARLOTTE A. Note on Regular Polygons. Read Oct. 27, 1906. *Annals of Mathematics*, ser. 2, vol. 8, No. 3, pp. 127–134 ; Apr., 1907.

SEE, T. J. J. On the Physical State of the Matter of the Earth's Interior, with Considerations on Terrestrial Geology, and on the Comparative Geology of the Other Planets. Read (San Francisco) Feb. 25, 1905. *Astronomische Nachrichten*, vol. 169, No. 4053, pp. 321–364 ; Oct., 1905.

SISAM, C. H. On Septic Scrolls. Read Dec. 29, 1904. *American Journal of Mathematics*, vol. 29, No. 1, pp. 48–100 ; Jan., 1907.

SMITH, CLARA E. A Theorem of Abel and its Application to the Development of an Arbitrary Function in Terms of Bessel's Functions. Read Dec. 29, 1905. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 92–106 ; Jan., 1907.

—— Development of a Function in Terms of Bessel's Functions (Second Paper). Read Feb. 24, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 92–106 ; Jan., 1907.

SNYDER, V. On a Type of Rational Twisted Curves. Read Sept. 8, 1905, *American Journal of Mathematics*, vol. 28, No. 3, pp. 237–242 ; July. 1906.

—— On Twisted Curves Contained in a Linear Complex. Read April 28, 1906. *American Journal of Mathematics*, vol. 29, No. 3, pp. 279–288; July, 1907.

STICKELBERGER, L. Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören. Read Sept. 3, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 4, pp. 509–513; Oct., 1906.

STORY, W. E. Denumerants of Double Differentiants. Read Dec. 29, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 1, pp. 33–70; Jan., 1907.

VAN VLECK, E. B. A Proof of Some Theorems on Pointwise Discontinuous Functions. Read Oct. 29, 1904. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 189–204; Apr., 1907.

—— Some Theorems of Pointwise Discontinuous Functions; Supplementary Note. Read Apr. 29, 1905. Published with the previous paper.

WHITE, H. S. Poncelet Quadrilaterals on a Curve of the Third Order and a Conic. Read Oct. 28, 1905. *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 172–176; July, 1906.

WILCZYNSKI, E. J. Projective Differential Geometry of Plane Curves. Read (San Francisco) Feb. 25, 1905. Included in Chapter 3 of the Author's Projective Differential Geometry of Curves and Ruled Surfaces, Leipzig, 1906.

—— Projective Differential Geometry. Read Apr. 29, 1905. Incorporated in the paper of Feb. 24, 1906 (see below).

—— On a System of Partial Differential Equations in Involution. Read (San Francisco) Sept. 30, 1905. Incorporated in the paper of Feb. 24, 1906 (see below).

—— Outline of a Differential Projective Geometry of Curved Surfaces. Read (San Francisco) Feb. 24, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 2, pp. 233–260; Apr., 1907.

—— Projective Differential Geometry. Four Lectures. Read Sept. 5–8, 1906. *Bulletin of the American Mathematical Society*, vol. 13, No. 3, pp. 102–105; Dec., 1906.

WILSON, E. B. On Divergence and Curl. Read Sept. 4, 1906. *American Journal of Science*, ser. 4, vol. 23, No. 135, pp. 214–220; Mar., 1907.

—— Oblique Reflections and Unimodular Strains. Read Sept. 4, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 2. pp. 270–298; Apr., 1907. See also *Annals of Mathematics*, ser. 2, vol. 8, No. 2, pp. 77–86; Jan., 1907.

—— On the Revolutions of a Dark Body about the Sun. Read Feb. 23, 1907. *Annals of Mathematics*, ser. 2, vol. 8, pp. 135–148; Apr., 1907.

WRIGHT, J. E. Correspondences and the Theory of Continuous Groups Read Dec. 29, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 3, pp. 391–400; July, 1906.

—— Arrangement of Ovals of a Plane Sextic Curve. Read Apr. 27, 1907. *American Journal of Mathematics*, vol. 29, No. 3, pp. 305–308; July, 1907.

YOUNG, J. W. On a Class of Discontinuous ζ-Groups Defined by Normal Curves of the Fourth Order in a Space of four Dimensions. Read (Chicago) Dec. 29, 1905. *Rendiconti del Circolo Matematico di Palermo*, vol. 23, No. 1, pp. 97–106; Jan.–Feb., 1907.

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